The fractional - order controllers: Methods for their synthesis and application

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Abstract

This paper deals with fractional-order controllers. We outline mathematical description of fractional controllers and methods of their synthesis and application. Synthesis method is a modified root locus method for fractional-order systems and fractional-order controllers. In the next section we describe how to apply the fractional controller on control systems.

Keywords: fractional-order controller, fractional-order controlled system, controller synthesis, control algorithm

1 Introduction

The real objects are generally fractional [17], however, for many of them the fractionality is very low. A typical example of a non-integer (fractional) order system is the voltage-current relation of a semi-infinite lossy RC line or diffusion of the heat into a semi-infinite solid, where heat flow is equal to the half-derivative of temperature [16]:

$$\frac{d^{0.5}T(t)}{dt^{0.5}} = q(t).$$

Their integer-order description can cause significant differences between mathematical model and the real system. The main reason for using integer-order models was the absence of solution methods for fractional-order differential equations. Recently, important achievements [1, 3, 6, 8, 13, 15, 18, 19] were obtained, which enable to take into account real order of dynamic systems.

PID controllers belong to dominating industrial controllers and therefore are objects of steady effort for improvements of their quality and robustness. One of the possibilities to improve PID controllers is to use fractional-order controllers with non-integer derivation and integration parts. For fractional-order systems the fractional controller $CRONE$ [9] has been developed, PD^{δ} controller [3] and the $PI^{\lambda}D^{\delta}$ controller [12, 14] has been suggested.

Before the actual design of controllers for dynamical systems it is necessary to identify these systems [2, 15], and then determine their dynamical properties. The dynamical properties of the system under observation can be expressed mathematically and graphically in the form of various characteristics. These characteristics can be determined from the differential equation or transfer function via computation or experimentally by exciting the system from the equilibrium. Computation of the transfer characteristics of the fractional-order dynamical systems has been the subject of several publications, e.g. by numerical methods [3], as well as analytical methods [13]. In the synthesis of a controller its parameters are determined according to the given requirements. These requirements are, for example, the stability measure, the accuracy of the regulation process, dynamical properties etc. The check whether the requirements have been met can be done with a simulation on the control circuit model. There are a large number of methods for the design of integer-order controllers, but the situation is worse in the case of fractional-order controllers where the methods are only being worked out. One of the methods being developed is the method (modification of roots locus method) of dominant roots [10], based on the given stability measure and the damping measure of the control circuit.

2 Basic mathematical tools for fractional calculus

The fractional calculus is a generalization of integration and derivation to non-integer order operators. The idea of fractional calculus has been known since the development of the regular calculus, with the first reference probably being associated with Leibniz and L'Hospital in 1695. At first, we generalize the differential and integral operators into one fundamental operator D_t^{α} which is known as fractional calculus:

$$_aD_t^{\alpha} = \begin{cases} \frac{d^{\alpha}}{dt^{\alpha}} & \Re(\alpha) > 0, \\ 1 & \Re(\alpha) = 0, \\ \int_a^t (d\tau)^{-\alpha} & \Re(\alpha) < 0. \end{cases}$$

The two definitions used for the general fractional differintegral are the Grünwald definition and

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the Riemann-Liouville (RL) definition [8]. The Grünwald definition is given here

$$ {}_aD_t^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{j=0}^{[\frac{t-a}{h}]} (-1)^j \binom{\alpha}{j} f(t-jh), \quad (1) $$

where $[x]$ means the integer part of x. The RL definition is given as

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad (2) $$

for $(n-1 < \alpha < n)$ and where $\Gamma(x)$ is the well known Euler's *gamma* function.

The Laplace transform method is used for solving engineering problems. The formula for the Laplace transform of the RL fractional derivative (2) has the form [13]:

$$ \int_0^\infty e^{-pt}\, {}_0D_t^\alpha f(t)\, dt = $$

$$ = p^\alpha F(p) - \sum_{k=0}^{n-1} p^k \; {}_0D_t^{\alpha-k-1} f(t)\big|_{t=0}, \quad (3) $$

for $(n-1 < \alpha \le n)$.

For numerical calculation of fractional-order derivation we can use the relation (4) derived from the Grünwald definition (1). This relation has the following form:

$$ {}_{(t-L)}D_t^\alpha f(t) \approx h^{-\alpha} \sum_{j=0}^{N(t)} b_j f(t-jh), \quad (4) $$

where L is the "memory length", h is the step size of the calculation,

$$ N(t) = \min\left\{ \left[\frac{t}{h}\right], \left[\frac{L}{h}\right] \right\}, $$

[x] is the integer part of x and b_j is the binomial coefficient:

$$ b_0 = 1, \qquad b_j = \left(1 - \frac{1+\alpha}{j}\right) b_{j-1}. \quad (5) $$

For the solution of the fractional-order differential equations (FODE) most effective and easy analytic methods were developed based on the formula of the Laplace transform method of the Mittag-Leffler function in two parameters [13]. A two-parameter function of the Mittag-Leffler type is defined by the series expansion:

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \quad (\alpha,\beta > 0). \quad (6) $$

The Mittag-Leffler function is a generalization of exponential function e^z and the exponential function is a particular case of the Mittag-Leffler function. Here is the relationship given in [13]:

$$ E_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z. $$

3 Fractional - order closed control loop

We will be studying feed-back control system with unit gain in the feed-back loop (Fig.1), where $G_r(p)$ is the controller transfer function, $G_s(p)$ is controlled system transfer function, $W(p)$ is an input, $E(p)$ is an error, $U(p)$ is output from controller and $Y(p)$ is output from system. The transfer function of close

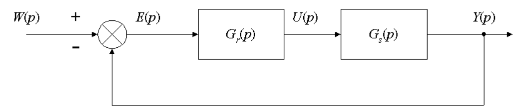

Fig.1: *Feed - back control circuit*

feed-back control circuit (Fig.1) has the form:

$$ G_c(p) = \frac{Y(p)}{W(p)} = \frac{G_r(p)G_s(p)}{1+G_r(p)G_s(p)}. \quad (7) $$

3.1 Fractional - order controlled systems

The fractional-order controlled system will be represented with fractional model with transfer function given by the following expression [14]:

$$ G_s(p) = \frac{Y(p)}{U(p)} = \frac{1}{a_n p^{\beta_n} + \ldots + a_1 p^{\beta_1} + a_0 p^{\beta_0}}, \quad (8) $$

where β_k, $(k = 0, 1, 2, \ldots, n)$ are generally real numbers, $\beta_n > \beta_{n-1} > \ldots > \beta_1 > \beta_0 \ge 0$ and a_k $(k = 0, 1, \ldots, n)$ are arbitrary constants. In the time domain, the transfer function (8) corresponds to the n-term FODE with constant coefficients

$$ a_n\, D_t^{\beta_n} y(t) + \ldots + a_1\, D_t^{\beta_1} y(t) + a_0\, D_t^{\beta_0} y(t) = u(t). \quad (9) $$

Identification methods [2, 15] for determination of the coefficients a_k and β_k, $(k = 0, 1, \ldots, n)$ were developed, based on minimization of the difference between the calculated (y^c) and experimentally measured (y^e) values

$$ Q = \frac{1}{M+1} \sum_{m=0}^{M} [y_m^e - y_m^c]^2, $$

where M is the number of measured values.

For the analytical solution of the n-term FODE (9) we can write formula in general form [13]:

$$ y(t) = $$

$$ = \frac{1}{a_n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{\substack{k_0+k_1+\ldots+k_{n-2}=m \\ k_0\ge 0;\ldots,k_{n-2}\ge 0}} (m; k_0, k_1, \ldots, k_{n-2}) $$

$$\prod_{i=0}^{n-2}\left(\frac{a_i}{a_n}\right)^{k_i} t^{(\beta_n-\beta_{n-1})m+\beta_n+\sum_{j=0}^{n-2}(\beta_{n-1}-\beta_j)k_j-1}$$

$$E^{(m)}_{\beta_n-\beta_{n-1},+\beta_n+\sum_{j=0}^{n-2}(\beta_{n-1}-\beta_j)k_j}\left(-\frac{a_{n-1}}{a_n}t^{\beta_n-\beta_{n-1}}\right),$$

where $E_{\lambda,\mu}(z)$ is the Mittag-Leffler function in two parameters (6),

$$E^{(n)}_{\lambda,\mu}(y) \equiv \frac{d^n}{dy^n}E_{\lambda,\mu}(y) = \sum_{j=0}^{\infty}\frac{(j+n)!\ y^j}{j!\ \Gamma(\lambda j+\lambda n+\mu)},$$

for $(n = 0, 1, 2, ...)$.

For the numerical solution of the n-term FODE (9) we can write formula in general form:

$$y(k) = \frac{u(k) - \sum_{i=1}^{n}(a_i h^{-\beta_i}\sum_{j=1}^{k} b_j y(k-j))}{\sum_{i=0}^{n} a_i h^{-\beta_i} b_0},$$

for $k = 1, 2, 3, \ldots$, $y(0) = 0$, $y(1) = 0$, where $u(k)$ is a function on the right side of the differential equation.

3.2 Fractional - order controllers

The fractional-order controller will be represented by fractional-order $PI^\lambda D^\delta$ controller with transfer function given by the following expression [14]:

$$G_r(p) = \frac{U(p)}{E(p)} = K + T_i p^{-\lambda} + T_d p^\delta, \qquad (10)$$

where λ and δ are an arbitrary real numbers ($\lambda, \delta \geq 0$), K is amplification (gain), T_i is integration constant and T_d is differentiation constant. In the time domain equation (10) has the form:

$$u(t) = Ke(t) + T_i D_t^{-\lambda} e(t) + T_d D_t^\delta e(t). \qquad (11)$$

Taking $\lambda = 1$ and $\delta = 1$, we obtain a classical PID controller. If $\lambda = 0$ ($T_i = 0$) we obtain a PD^δ controller, etc. All these types of controllers are particular cases of the $PI^\lambda D^\delta$ controller. The $PI^\lambda D^\delta$ controller (11) is more flexible and gives an opportunity to better adjust the dynamical properties of the fractional-order control system.

4 Synthesis of fractional - order controllers

For the design of fractional-order controller a new method was suggested based on the dominant roots principle. This method is based on from poles distribution of the characteristic equation in the complex plane (Fig.2).

Values of dominant roots are designed for the quality requirement of the control circuit. Their significance is, that dominant roots are defined for stability measure S_t and damping measure T_l. Under there conditions the complex conjugate roots satisfy the equation:

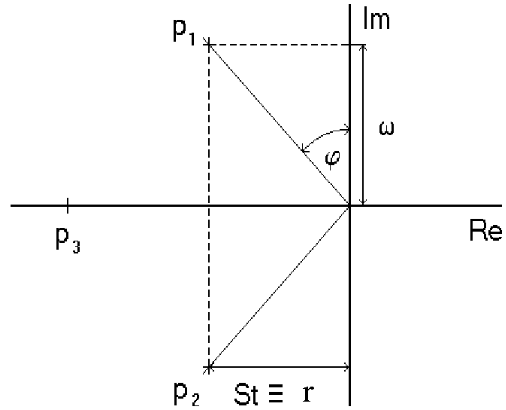

Fig.2: *Roots in the complex plane*

$$p_{1,2} = -r \pm i\omega. \qquad (12)$$

The parameters of controller were set up so that other poles were from dominant remote to the left side. The parameters design of the fractional-order controller can be divided into two stages:

1. **Design of parameter K**

   Proportional parameter K influence the value of static deviation E_t [%], control time T_r [s], and overshoot P_r [%]. Generally, with increased value of K, control time T_r [s] is decreasing and static deviation E_t [%] is lower:

   $$K \geq (100/E_t) - a_0.$$

2. **Design of parameters T_d, δ, T_i, λ**

   We define the required stability measure $S_t = r$ and damping measure $T_l = \frac{r}{\omega}$. This requirement is satisfied by the complex conjugate roots (12). We will similarly use characteristic equation as the classical root locus method [4]. The characteristic equation of fractional-order control loop (7) has the form:

   $$G_r(p)G_s(p) + 1 = 0. \qquad (13)$$

   After substitution of the fractional-order controller transfer function (10) and the fractional-order controlled system transfer function (8) and after some manipulation we obtain characteristic equation (13) in the following form:

   $$\sum_{k=0}^{n} a_k p^{\beta_k} + (K + T_i p^{-\lambda} + T_d p^\delta) = 0. \qquad (14)$$

This algebraic equation is in the general form for the n-term fractional-order controlled system and fractional-order $PI^{\lambda}D^{\delta}$ controller. Solution of this equation for known poles and parameter K gives unknown parameters T_d, δ, T_i, λ. The algebraic equation (14) is solved in the space of complex variable. Result are parameters of fractional-order controller for required stability measure and damping measure.

5 Control algorithm for fractional - order controllers

The control algorithm was designed according to the control scheme in Fig.1. The position algorithm uses discrete time steps k and consists of the following steps [11]:

1. **Filtering of required value:**

$$w^*(k) = w^*(k-1) + 0.5(w(k) - w^*(k-1)),$$

where $w(k)$ is required value.

2. **Calculating the control error:**

$$e(k) = w^*(k) - y(k),$$

for discrete time step $(k = 1, 2, \ldots)$, where $e(k)$ is regulation error and $y(k)$ is measured value.

3. **Determination of control value:**

$$u(k) = K\ e(k) + \frac{T_i}{T^{-\lambda}} \sum_{j=v}^{k} q_j e(k-j) + + \frac{T_d}{T^{\delta}} \sum_{j=v}^{k} d_j e(k-j),$$

for discrete time step $(k = 1, 2, \ldots)$, where T is the length of time step (sample period). The binomial coefficients d_j and q_j were calculated from the recurrent equation (5). The numerical algorithm (4) requires to store the whole history $(v = 0)$. For improving their effectiveness we have used "short memory" principle [3], where $v = 0$ for $k < (L/T)$ or $v = k - (L/T)$ for $k > (L/T)$. Besides the "short memory" the control quality is influenced by time step T.

Fractional-order controllers can be realized as software or by passive or active electrical elements [5, 7].

6 Example

We verify the above methods on an example from [3]. Assume the system which we can describe by three-term $(n = 2)$ differential equation with coefficients $a_2 = 0.8, a_1 = 0.5, a_0 = 1, \beta_2 = 2.2, \beta_1 = 0.9, \beta_0 = 0$. After its approximation with an integer-order system we have $a_2 = 0.7414, a_1 = 0.2313, a_0 = 1, \beta_2 = 2, \beta_1 = 1, \beta_0 = 0$. The integer-order PD controller designed with the dominant roots method applied to the approximated system has the parameters $K = 20.5, T_d = 2.7343, \delta = 1$. By applying the controller to the original system we do not achieve the required quality of the regulation process as with the approximated integer-order system. This was confirmed via simulation in the time domain [3] and also by checking the stability measure and damping measure.

This proves the inadequacy of approximating the fractional-order system with an integer-order system for the purpose of controller design. It is suitable to consider fractional-order systems and also controllers, by means of which it is possible to obtain higher quality regulation and robustness. For the fractional-order PD^{δ} controller we then have $K = 20.5, T_d = 5.79, \delta = 0.95$ and the required quality of regulation is ensured also in the original fractional-order system [10].

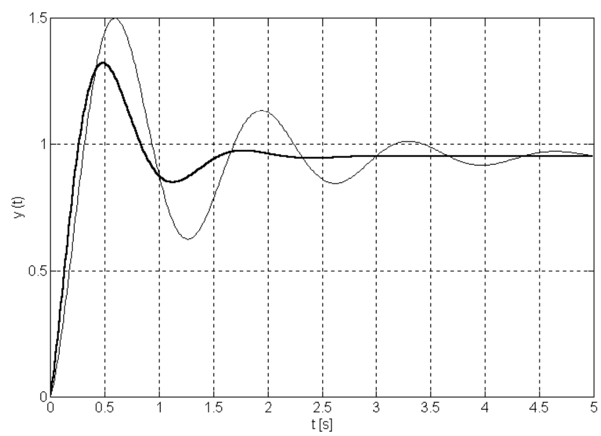

Fig.3: *Transient unit - step responses*

On the Fig.3 is showed the comparison of the unit-step responses the integer-order PD controller applied on the fractional-order system (thin line) and the fractional-order PD^{δ} controller applied on the fractional-order system (thick line).

The dynamical properties of the closed loop with fractional-order controlled system and the fractional-order controller are better than the dynamical properties of the closed loop with the integer-order controller. The systems with the integer-order controller stabilizes slower and has larger surplus oscillations. We can see that use of the fractional-order controller leads to the improvement of the control of the fractional-order system.

7 Conclusion

The above methods make it possible to design fractional-order controllers with given measures of stability and damping. The results of previous works also show that fractional-order controllers are more robust [12], which means they are less sensitive to changes of the system parameters and controller parameters. This can even lead to qualitatively different dynamical phenomena in control circuits.

Acknowledgement

The author thanks to Dr. Lubomir Dorcak for discussion of this problem and also to Dr. Ladislav Pivka for checking the language of this paper. This work was supported by grant VEGA 1/4333/97 from the Slovak Agency for Science.

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