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Fuzzy goal programming – A parametric approach

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ABSTRACT

Narasimhan incorporated fuzzy set theory within goal formulation in 1980. Since then, much research has been performed in this field, and various models for solving fuzzy goal programming have been proposed. One of the well-known models was proposed by Tiwari et al. in 1987 [19], where an additive model was proposed. This paper is an extension to the Tiwari et al. model that deals with the sum of weighted negative deviations between the desirable achievement degree and the common target. Here, properties of the model are proposed. A numerical example is also given to illustrate the approach.

1. Introduction

Goal programming (GP) [15,17,18] is useful for decision makers to consider simultaneously several objectives in finding a set of acceptable solutions. However, determining precisely the goal value of each objective is difficult for decision makers as it is possible that only partial information can be obtained [23]. To incorporate uncertainty and imprecision into the formulation, some approaches have been used to reformulate the GP models, such as using a probability distribution, a penalty function and various types of thresholds [1,6,10,14].

To specify imprecise aspiration levels of the goals in a fuzzy environment, fuzzy set theory [22] has been introduced in the field.

Narasimhan [13] had initially proposed fuzzy goal programming (FGP) by using membership functions. His work is inspired by the fuzzy programming approach introduced by Zimmermann [24]. Some researchers have provided further investigation of FGP with respect to problem formulation, the relative importance, and the fuzzy priority of fuzzy goals. Most previous researchers, except for Tiwari et al. [19], have used the min operator to find the fuzzy decision that simultaneously satisfies fuzzy goals and fuzzy constraints. Although this approach is computationally efficient, its application may produce "uniform" membership degrees for fuzzy goals when the achievement of some goals is stringently required. Tiwari et al. [19] have used an additive model to manage FGP, which includes two cases: (1) the simple additive model, which takes the sum of each goal's achievement degree, and (2) the weighted additive model, which uses different weights for the various goals to reflect the relative importance of the goals and then takes the sum of each goal's achievement degree with the weights as the coefficients. Chen and Tsai [2] proposed an extension of the additive model to consider goals of different importance where the relative importance of the goals is reflected by the corresponding desirable achievement degrees. There are some other fuzzy goal programming models [8,21]. Although those models constitute a flexible tool for solving fuzzy goal programming,

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it is hard for decision makers to make a choice with this tool in a practical problem. Thus, it is important to propose a unified fuzzy goal programming model comprising all of the previous instances as specific cases.

In this paper, a unified fuzzy goal programming model is proposed based on a parametric approach. The new model is an extension of the additive model, which incorporates the different importance of goals and common targets for the desirable achievement degrees of fuzzy goals. With the common target taken as a parameter, our new model is a parameter problem. The FGP models mentioned above in the last paragraph are all special cases where our new model can be employed. Some properties of our new model are also proposed.

2. Related models

In conventional GP models, decision makers are required to specify a precise aspiration level for each of the objectives. In general, especially in large scale problems, this can be quite a difficult task for decision makers. Applying fuzzy set theory to GP has the advantage that decision makers are allowed to specify imprecise aspiration levels. Here, an objective with an imprecise aspiration level can be treated as a fuzzy goal. In this paper, we will consider the following FGP problem, which contains m fuzzy goals $G_i(x)$:

$$\begin{aligned}&G_i(x) \gtrsim g_i \quad (\text{or} \quad G_i(x) \lesssim g_i) \quad i = 1, 2, \ldots, m\\&\text{subject to} \quad Ax \leqslant b; \quad x \geqslant 0\end{aligned} \tag{1}$$

where $G_i(x) \gtrsim (\lesssim) g_i$ indicates the fuzzy goal approximately greater than or equal to (approximately less than or equal to) the aspiration level g_i.

Furthermore, we take the following assumption:

(A) $G_i(x)$, $i = 1, 2, \ldots, m$, are continuous functions;
(B) Let $S = \{x|Ax \leqslant b, x \geqslant 0\}$ be a non-empty compact set.

The fuzzy goals can be identified as fuzzy sets defined over a feasible set with membership functions. The linear membership function μ_i for the ith fuzzy goal $G_i(x) \gtrsim g_i$ can be expressed as

$$\mu_i(x) = \begin{cases} 1, & G_i(x) \geqslant g_i; \\ 1 - \frac{g_i - G_i(x)}{\Delta_{iL}}, & g_i - \Delta_{iL} \leqslant G_i(x) \leqslant g_i; \\ 0, & G_i(x) \leqslant g_i - \Delta_{iL} \end{cases} \tag{2.a}$$

where Δ_{iL} is the lower maximum admissible violation from the aspiration level g_i.

In case of the fuzzy goal $G_i(x) \lesssim g_i$, the membership function is defined as

$$\mu_i(x) = \begin{cases} 1, & G_i(x) \leqslant g_i; \\ 1 - \frac{G_i(x) - g_i}{\Delta_{iR}}, & g_i \leqslant G_i(x) \leqslant g_i + \Delta_{iR}; \\ 0, & G_i(x) \geqslant g_i + \Delta_{iR} \end{cases} \tag{2.b}$$

where Δ_{iR} are chosen constants of the upper maximum admissible violations from the aspiration level g_i. The constants Δ_{iL} and Δ_{iR} are either subjectively chosen by decision makers [13] or are tolerances in a technical process [9].

Thus, the problem (1) can be transformed into the multi-objective model:

$$\begin{aligned}&max \quad \{\mu_i(x), \mu_2(x), \ldots, \mu_m(x)\}\\&s.t. \quad x \in S = \{x|Ax \leqslant b, x \geqslant 0\}\\&\qquad 0 \leqslant \mu_i(x) \leqslant 1, \quad i = 1, 2, \ldots, m\end{aligned} \tag{3}$$

The first attempt for solving FGP problems was undertaken by Narasimhan [13]. Hannan [4,5] introduced conventional deviation variables into the model so that only a conventional linear programming formulation is required. Yang et al. [20] proposed the efficiency computation method and extended the well-known Zimmermann's approach [24] to transform the problem into a single objective linear programming model, as follows:

$$\begin{aligned}&max \quad \lambda\\&s.t. \quad x \in S = \{x|Ax \leqslant b, x \geqslant 0\}\\&\qquad \lambda \leqslant \mu_i(x) \leqslant 1, \quad i = 1, 2, \ldots, m\\&\qquad \lambda \geqslant 0\end{aligned} \tag{4}$$

The model above uses the min-operator for aggregating goals to determine the decision set and then to find the element with the highest membership degree. Thus, problem (4) can be represented in

$$\begin{array}{lll} \max\limits_{x} \min\limits_{1 \leqslant k \leqslant m} & \mu_k(x) \\ s.t. & x \in S = \{x | Ax \leqslant b, x \geqslant 0\} \\ & 0 \leqslant \mu_i(x) \leqslant 1, \quad i = 1, 2, \ldots, m \end{array} \tag{5}$$

i.e.,

$$\max_{x \in XS} \min_{1 \leqslant k \leqslant m} \mu_k(x) \tag{6}$$

where $XS = S \bigcap \{x | 0 \leqslant \mu_i(x) \leqslant 1, i = 1, 2, \ldots, m\}$. By assumption (B), we can know that the set XS is a compact set.

Instead, Tiwari et al. [19] proposed a simple additive model to formulate a FGP problem. The simple additive model of the FGP problem (1) is formulated by adding the membership functions together as

$$\begin{array}{ll} \max & \sum\limits_{k=1}^{m} \mu_k(x) \\ s.t. & x \in S = \{x | Ax \leqslant b, x \geqslant 0\} \\ & 0 \leqslant \mu_i(x) \leqslant 1, \quad i = 1, 2, \ldots, m \end{array}$$

i.e.,

$$\max_{x \in XS} \sum_{k=1}^{m} \mu_k(x) \tag{7}$$

The problem in (7) is to be optimized by maximizing the sum of each goal's achievement degree $\mu_k(x)$. The achievement degrees of some goals will not decrease because of a specific goal that is difficult to achieve, while the achievement of all goals will be lower if the min-operator is used for the same conditions [2] as in problem (6).

To reflect the relative importance of the goals, Tiwari et al. [19] further proposed a weighted additive model. Decision makers assign different weights as coefficients of the individual terms in the simple additive fuzzy achievement function to reflect their relative importance, i.e., the objective function is formulated by multiplying each membership of the fuzzy goal with a suitable weight and then adding them together. This procedure leads us to the following formulation:

$$\max_{x \in XS} \sum_{k=1}^{m} w_k \mu_k(x) \tag{8}$$

where w_k denotes the weight of the kth fuzzy goal and $\sum_{k=1}^{m} w_k = 1, w_k \geqslant 0, \ k = 1, 2, \ldots, m.$

There are many methods to assess these weights. We may mention in this regard the eigenvector method of Satty [16], a geometric averaging procedure for constructing super-transitive approximations to binary comparison matrices by Narasimhan [12], the entropy method of Jaynes [7] and the weighted least squares method of Chu et al. [3]. These methods can be used to suitably specify the weights. Weights in the weighted additive model reveal the relative importance of the fuzzy goals. Remark (i): It is relatively simple to recognize problem (7) as a special case of problem (8) with the weights $w_k = \frac{1}{m}$. Remark (ii): We may take the convex combination of problems (6) and (8) to balance the two different aggregate operators

$$\max_{x \in XS} \left[(1 - t) \sum_{k=1}^{m} w_k \mu_k(x) + t \min_{1 \leqslant k \leqslant m} \mu_k(x) \right] \tag{9}$$

where $t \in [0, 1]$ reflects the preference of decision makers. It is easy to observe that problem (9) has, as specific cases, (6) and (8), when t is equal to 0 and 1, respectively. However, it is not an easy task to find real meaning for the parameter t to be chosen in (9). Remark (iii): Model (8) may produce undesirable solutions when the weights are changed [2] (see the numerical example below in subsection 3.3).

A new model is proposed by Chen and Tsai [2], which allows decision makers to determine explicitly a desirable achievement degree for each fuzzy goal as the importance of the fuzzy goal, i.e.,

$$\begin{array}{ll} \max & \sum\limits_{k=1}^{m} \mu_k(x) \\ s.t. & x \in S = \{x | Ax \leqslant b, x \geqslant 0\} \\ & 0 \leqslant \mu_i(x) \leqslant 1, \quad i = 1, 2, \ldots, m \\ & \mu_i(x) \geqslant z_i, \quad i = 1, 2, \ldots, m \end{array} \tag{10}$$

where z_i denotes the desirable achievement degree specified by decision makers.

However, the determination of a desirable achievement degree for each fuzzy goal may be a difficult task for decision makers.

3. A new FGP model – a parametric approach

3.1. The new FGP model

In this subsection, a unified FGP model is proposed based on a parametric approach. Suppose we could give, for all of the membership functions $\mu_i(x)$, a common target $\mu_i(x) \geqslant z$, for all $i = 1, 2, \ldots, m$. This construct is possible because all membership functions are in the unit interval [0,1]. We then get the following goal programming formulation:

$$\begin{aligned} max \quad & \sum_{k=1}^{m} w_k min(\mu_k(x) - z, 0) \\ s.t. \quad & x \in S = \{x | Ax \leqslant b, x \geqslant 0\} \\ & 0 \leqslant \mu_i(x) \leqslant 1, \quad i = 1, 2, \ldots, m \end{aligned} \tag{11}$$

In fact, if we take the sum of the weighted negative deviation between $\mu_k(x)$ and target z as the objective function, then (11) can be rewritten as

$$\max_{x \in XS} \sum_{k=1}^{m} w_k \{min(\mu_k(x), z) - z\} \tag{12}$$

Thus, for each z, the problem

$$\max_{x \in XS} \sum_{k=1}^{m} w_k min(\mu_k(x), z) \tag{13}$$

has the same optimal solution as problem (11).

Remark (i): The target $z \in [0,1]$ may be specified by decision makers. Some earlier models [11,24] took the value of $z = 1$. In real decision making, however, every decision maker may have a different choice. If we take z as a parameter, then (13) becomes a parametric problem, which is our new FGP model.

Remark (ii): The weights w_k, $k = 1, 2, \ldots, m$, represent the relative importance of each fuzzy goal. Their values can be specified by decision makers or can be determined by other well-known methods [3,7,12,16]. In this paper, we assume that the values are equivalent to the weights of (8).

Denoted by z_{maxmax} and z_{maxmin}, respectively the optimal value of the following problem

$$z_{maxmax} = \max_{x \in XS} \max_{1 \leqslant k \leqslant m} \quad \mu_k(x)$$
$$z_{maxmin} = \max_{x \in XS} \min_{1 \leqslant k \leqslant m} \quad \mu_k(x)$$

which, because of the compactness of XS and the continuity of the membership function $\mu_k(x)$, are finite and can be attained.

3.2. The properties of the new model

In this subsection, the relationships between our new model (13), in conjunction with the models in Section 2, are given.

Proposition 1. *For any $z \geqslant z_{maxmax}$, (13) is equivalent to (8).*

Proof. Because for $z \geqslant z_{maxmax}$, one has $\mu_k(x) \leqslant z$, $k = 1, 2, \ldots, m$. Therefore

$$\sum_{k=1}^{m} w_k min(\mu_k(x), z) = \sum_{k=1}^{m} w_k \mu_k(x)$$

for each $x \in XS$. □

Proposition 2. *For $z = z_{maxmin}$, (13) is equivalent to (6).*

Proof. Let x_{maxmin} be an optimal solution to (6) so that

$$\mu_k(x_{maxmin}) \geqslant z_{maxmin} \tag{14}$$

for each $k = 1, 2, \ldots, m$.

Thus, for any $x \in XS$, using (14) we have

$$\sum_{k=1}^{m} w_k min(\mu_k(x), z_{maxmin}) \leqslant \sum_{k=1}^{m} w_k z_{maxmin} = \sum_{k=1}^{m} w_k min(\mu_k(x_{maxmin}), z_{maxmin})$$

Thus, x_{maxmin} is an optimal solution to (13) with parameter $z = z_{maxmin}$.

Conversely, let x^* be an optimal solution to (13) with $z = z_{maxmin}$. By the optimality of x^* and (14), we can get

$$\sum_{k=1}^{m} w_k z_{maxmin} \geqslant \sum_{k=1}^{m} w_k min(\mu_k(x^*), z_{maxmin}) \geqslant \sum_{k=1}^{m} w_k min(\mu_k(x_{maxmin}), z_{maxmin}) = \sum_{k=1}^{m} w_k z_{maxmin}$$

Thus,

$$\begin{aligned} &min(\mu_k(x^*), z_{maxmin}) \geqslant z_{maxmin} \quad \text{for each} \quad k = 1, 2, \ldots, m \\ &\min_{1 \leqslant k \leqslant m} \mu_k(x^*) \geqslant z_{maxmin} \end{aligned}$$

Thus, x^* is also an optimal solution to problem (6).

By Propositions 1 and 2, we can determine that (8) and (6) are specific cases of (13). □

Proposition 3. *Let x^* be an arbitrary optimal solution to (13) with the parameter $z = z^*$, Then, any optimal solution for*

$$\begin{aligned} \max \quad & \sum_{k=1}^{m} \mu_k(x) \\ s.t. \quad & x \in S = \{x | Ax \leqslant b, x \geqslant 0\} \\ & 0 \leqslant \mu_i(x) \leqslant 1, \quad i = 1, 2, \ldots, m \\ & \mu_i(x) \geqslant \mu_i(x^*), \quad i = 1, 2, \ldots, m \end{aligned} \tag{15}$$

is optimal for (13) with $z = z^*$.

Proof. Because the set XS is compact and $\mu_k(x)$ is a continuous function, there exists an optimal solution x^* for (13) with arbitrary $z = z^*$.

The feasible region $F(x^*)$ of problem (15) is a non-empty compact set. For any point $x \in F(x^*)$, we have $\mu_k(x) \geqslant \mu_k(x^*)$.

$$w_k min(\mu_k(x), z^*) \geqslant w_k min(\mu_k(x^*), z^*)$$

Thus,

$$\sum_{k=1}^{m} w_k min(\mu_k(x), z^*) \geqslant \sum_{k=1}^{m} w_k min(\mu_k(x^*), z^*)$$

Because x^* is an optimal solution of (13), x is also an optimal solution of (13) with $z = z^*$. Thus, every feasible solution of (15) is the optimal solution of (13). □

Proposition 4. *Let x^* be an optimal solution to (8).* Then x^* *is an optimal solution to (13), with parameter $z \leqslant \min_{1 \leqslant k \leqslant m} \mu_k(x^*)$.*

Proof. Because $z \leqslant \min_{1 \leqslant k \leqslant m} \mu_k(x^*)$, $\mu_k(x^*) \geqslant z$, for each $k = 1, 2, \ldots, m$. We get

$$\sum_{k=1}^{m} w_k min(\mu_k(x^*), z) = \sum_{k=1}^{m} w_k z.$$

For (13),

$$\max_{x \in XS} \sum_{k=1}^{m} w_k min(\mu_k(x), z) \leqslant \max_{x \in XS} \sum_{k=1}^{m} w_k \mu_k(x) = \sum_{k=1}^{m} w_k \mu_k(x^*) = \max_{x \in XS} \sum_{k=1}^{m} w_k min(\mu_k(x^*), z)$$

which implies that x^* is also an optimal solution to (13) with parameter $z \leqslant \min_{1 \leqslant k \leqslant m} \mu_k(x^*)$. □

Proposition 5. *Let x^* be an optimal solution to (8). Then x^* is an optimal solution to problem (13) with parameter $z \geqslant \max_{1 \leqslant k \leqslant m} \mu_k(x^*)$.*

Proof. Assume that x^* is not the optimal solution to problem (13) for any $z \geqslant \max_{1 \leqslant k \leqslant m} \mu_k(x^*)$; then, there exists $x \neq x^*$ such that

$$\sum_{k=1}^{m} w_k \mu_k(x) \geqslant \sum_{k=1}^{m} w_k min(\mu_k(x), z) > \sum_{k=1}^{m} w_k min(\mu_k(x^*), z) = \sum_{k=1}^{m} w_k \mu_k(x^*)$$

which is a contradiction because x^* is the optimal solution to (8).

Remark 1. By Propositions 1, 2 and 5, we can determine that the solution aset of (13) when z varies on the unit interval $[0,1]$ is the same as the solution set of (13) when z varies $[z_{maxmin}, \max_{1\leqslant k\leqslant m} \mu_k(x^*)]$. Furthermore, the relationship between (8) and (13) has been shown in Propositions 1, 4 and 5

The following Propositions (6 and 7) provide properties of our model that concern changing the parameter z and the weight vector. It is shown that our model still produces proper solutions when the weight vector is changed. □

Proposition 6. *Let x_* be an optimal solution of (13) with parameter $z = z_1$, and let x^* be an optimal solution of (13),* with parameter $z = z_2$. *If $z_1 < z_2$, then there at least exists $j_0 \in J = \{j|\mu_j(x_*) < z_2\}$ such that $\mu_{j_0}(x^*) \geqslant \mu_{j_0}(x_*)$.*

Proof. Assume that for all $j \in J = \{j|\mu_j(x_*) < z_2\}$, we have $\mu_j(x^*) < \mu_j(x_*)$. Because x_* is feasible for (13) with parameter $z = z_2$, we get

$$\sum_{k=1}^{m} w_k min(\mu_k(x^*), z_2) \geqslant \sum_{k=1}^{m} w_k min(\mu_k(x_*), z_2)$$

$$\sum_{j\in J} w_j\mu_j(x_*) + \sum_{j\in J^c} w_j min(\mu_j(x^*), z_2) > \sum_{k=1}^{m} w_k min(\mu_k(x^*), z_2) \geqslant \sum_{k=1}^{m} w_k min(\mu_k(x_*), z_2) = \sum_{j\in J} w_j\mu_j(x_*) + \sum_{j\in J^c} w_j z_2$$

where $J^c = \{1, 2, \ldots, m\}\backslash J$.
Then

$$\sum_{j\in J^c} w_j min(\mu_j(x^*), z_2) > \sum_{j\in J^c} w_j z_2$$

which is a contradiction. □

Remark 2. By Proposition 6, we can know that the achievement degree of an "inferior" fuzzy goal will not decrease through increasing the parameter value of z.

Proposition 7. *Let $W_* = (w_1, w_2, \ldots, w_m)$, $W^* = (w'_1, w'_2, \ldots, w'_m)$ be two different weight vectors, respectively, where $w'_{j_1} = w_{j_1} - \Delta$, $w'_{j_2} = w_{j_2} + \Delta$, $0 \leqslant \Delta$, $j_1, j_2 \in \{j|j = 1, 2, \ldots, m\}$, and $w'_j = w_j$ for all $j \in \{j|j = 1, 2, \ldots, m\}\backslash\{j_1, j_2\}$. Let x_*, x^* be the optimal solution to (13) with parameter z and weight vectors W_*, W^*, respectively.*

(i) If $\mu_{j_1}(x_*) < \mu_{j_2}(x_*)$, then we have $\mu_{j_1}(x^*) \leqslant \mu_{j_2}(x^*)$.
(ii) If $\mu_{j_1}(x_*) \geqslant z > \mu_{j_2}(x_*)$, then we have $\mu_{j_1}(x^*) < \mu_{j_1}(x_*)$ or $\mu_{j_2}(x^*) > \mu_{j_2}(x_*)$.

Proof. (i) There are three different cases:

(a) $z \leqslant \mu_{j_1}(x_*) < \mu_{j_2}(x_*)$
(b) $\mu_{j_1}(x_*) < \mu_{j_2}(x_*) \leqslant z$
(c) $\mu_{j_1}(x_*) < z < \mu_{j_2}(x_*)$

For case (a) $z \leqslant \mu_{j_1}(x_*) < \mu_{j_2}(x_*)$

$$\sum_{j=1}^{m} w_j min(\mu_j(x_*), z) = \sum_{j=1}^{m} w'_j min(\mu_j(x_*), z)$$
$$\sum_{j=1}^{m} w_j min(\mu_j(x^*), z) = \sum_{j=1}^{m} w'_j min(\mu_j(x^*), z) + \Delta(min(\mu_{j_1}(x^*), z) - min(\mu_{j_2}(x^*), z))$$

Because x_*,x^* is the optimal solution to (13) with weight vectors W_*,W^* respectively, we have

$$\Delta(min(\mu_{j_1}(x^*), z) - min(\mu_{j_2}(x^*), z)) < 0$$

Thus, $\mu_{j_1}(x^*) \leqslant \mu_{j_2}(x^*)$.

For cases (b) and (c), we can obtain the same result.

(ii) If $\mu_{j_1}(x_*) \geqslant z > \mu_{j_2}(x_*)$ then

Table 1
The results of different models.

Membership function	Model (6)	Model (7)	Model (8)	Model (10)	Model (13)
μ_1	0.8	0.981	1	0.991	0.9
μ_2	1	1	0.977	0.988	0.883
μ_3	0.741	0.605	0.636	0.621	0.9
μ_4	0.741	0.775	0.761	0.768	0.671
μ_5	0.974	0.965	0.939	0.952	0.834

$$\sum_{j=1}^{m} w_j min(\mu_j(x_*), z) = \sum_{j=1}^{m} w'_j min(\mu_j(x_*), z) + \Delta(z - \mu_{j_2}(x_*))$$
$$\sum_{j=1}^{m} w_j min(\mu_j(x^*), z) = \sum_{j=1}^{m} w'_j min(\mu_j(x^*), z) + \Delta(min(\mu_{j_1}(x^*), z) - min(\mu_{j_2}(x^*), z))$$

Because x_*,x^* is the optimal solution to (13) with weight vectors W_*, W^*, respectively, we have

$$z - \mu_{j_2}(x_*) > min(\mu_{j_1}(x^*), z) - min(\mu_{j_2}(x^*), z).$$

If $\mu_{j_1}(x^*) \geqslant z$ then we have

$$z - \mu_{j_2}(x_*) > z - min(\mu_{j_2}(x^*), z)$$

Thus,

$$min(\mu_{j_2}(x^*), z) > \mu_{j_2}(x_*)$$
$$\mu_{j_2}(x^*) > \mu_{j_2}(x_*)$$

If $\mu_{j_1}(x^*) < z$ then we have $\mu_{j_1}(x^*) < \mu_{j_1}(x_*)$. □

Remark 3. According to Proposition 7, the behavior of our model is not as good as model (10) when the weight vector is changed. However, it is shown that a change in the achievement degree of the fuzzy goal occurs when the weight vector is changed. This relationship will help decision makers make a proper choice for the weight vector. According to Propositions 6 and 7, if a decision maker wants to increase the achievement degree of a goal, then he can increase the value of z or the corresponding weight of the fuzzy goal.

3.3. Numerical example

As an illustration, Tiwari et al.'s example [19] containing five fuzzy goals with four variables and four system constraints is used here. The five fuzzy goals in the problem are described as

Table 2
The results of model (13) for different values of z.

Membership function	$z = 0.6$	$z = 0.741$	$z = 0.95$	$z = 1$
μ_1	0.8	0.8	0.95	1
μ_2	1	1	0.839	0.977
μ_3	0.741	0.741	0.95	0.636
μ_4	0.741	0.741	0.647	0.761
μ_5	0.974	0.974	0.78	0.939

Table 3
The results of models (13) and (8) with the weight vectors W1 = (0.49, 0.131, 0.153, 0.114, 0.112), W2 = (0.49, 0.131, 0.05, 0.217, 0112).

Membership function	(13) $z = 0.9$ Weight W1	(13) $z = 0.9$ Weight W2	(8) Weight W1	(8) Weight W2
μ_1	0.9	0.98	1	1
μ_2	0.883	1	0.977	0.977
μ_3	0.9	0.605	0.636	0.636
μ_4	0.671	0.775	0.761	0.761
μ_5	0.834	0.967	0.939	0.939

$$4x_1 + 2x_2 + 8x_3 + x_4 \lesssim 35$$
$$4x_1 + 7x_2 + 6x_3 + 2x_4 \gtrsim 100$$
$$x_1 - 6x_2 + 5x_3 + 10x_4 \gtrsim 120$$
$$5x_1 + 3x_2 + 2x_4 \gtrsim 70$$
$$4x_1 + 4x_2 + 4x_3 \lesssim 40$$

subject to

$$7x_1 + 5x_2 + 3x_3 + 2x_4 \leqslant 98$$
$$7x_1 + x_2 + 6x_3 + 6x_4 \leqslant 117$$
$$x_1 + x_2 + 2x_3 + 6x_4 \leqslant 130$$
$$9x_1 + x_2 + 6x_4 \leqslant 105$$
$$x_i \geqslant 0, \quad i = 1, 2, 3, 4$$

The tolerance and relative weights of the five fuzzy goals are (55, 40, 70, 30, 10) and (0.49, 0.131, 0.153, 0.114, 0.112), respectively. After formulating the fuzzy goals as membership functions by (2.a) and (2.b), we have the following results, which are given in Table 1 using different models.

For model (10), we suppose that the desirable achievement degrees for the five fuzzy goals are (0.8, 0.7, 0.5, 0.6, 0.8), respectively.

For model (13), we take the parameter z = 0.9. Thus, z_{maxmax} = 1, z_{maxmin} = 0.741.

If decision makers require or desire a very high achievement degrees (0.9, 0.9, 0.8, 0.9, 0.9) for each fuzzy goal in model (10), then such a model results in "no feasible solution".

However, for our new model (13), there always exists a feasible solution for all of the parameter values z. Furthermore, we can notice that the models (7), (8) and (10) give better achievement degrees for all of the goals compared to the proposed model (13), except for the third goal. However, the order of the achievement degree of the fuzzy goals in model (13) is more consistent with the order of weights.

Table 2 shows the results of model (13) for different values of z.

From Tables 1 and 2, we can determine that model (13) has the same solution as model (6) if $z \leqslant z_{maxmin}$ = 0.741. Model (13) has the same solution as model (8) if $z = z_{maxmax}$ = 1.

From Table 2, we also can know that the achievement degrees of the first and third goals increase if z varies from 0.741 to 0.95.

Table 3 shows the results of models (13) and (8) when the weights are changed.

From Table 3, we can know that the achievement degree of the fourth fuzzy goal increases (or the achievement degree of the third fuzzy goal decreases) if more emphasis is placed on the fourth fuzzy goal in model (13). However, the achievement degree of the fourth fuzzy goal does not change, even if more emphasis is placed on the fourth goal in model (8).

4. Conclusions

In this paper, a unified fuzzy goal programming model in a parametric form is proposed. The models [2,19,24] are special cases of the new model. Furthermore, our new model has several merits:

(i) Compared to the model proposed by Chen and Tsai [2], it is easy for decision makers to construct this model because they are required only to specify the common target. Some guidelines on the choice of the comment target are given in the Remarks of Propositions 5–7.
(ii) The model can still produce a "satisfying solution" when the weights are changed, i.e., according to Proposition 7, when the importance of some fuzzy goal increases, the achievement degree of the fuzzy goal will increase (or the achievement degree of other goals will decrease).

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