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# Extension of the TOPSIS method for decision making problems under interval-valued intuitionistic fuzzy environment ☆

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ABSTRACT

TOPSIS is one of the well-known methods for multiple attribute decision making (MADM). In this paper, we extend the TOPSIS method to solve multiple attribute group decision making (MAGDM) problems in interval-valued intuitionistic fuzzy environment in which all the preference information provided by the decision-makers is presented as interval-valued intuitionistic fuzzy decision matrices where each of the elements is characterized by interval-valued intuitionistic fuzzy number (IVIFNs), and the information about attribute weights is partially known. First, we use the interval-valued intuitionistic fuzzy hybrid geometric (IIFHG) operator to aggregate all individual interval-valued intuitionistic fuzzy decision matrices provided by the decision-makers into the collective interval-valued intuitionistic fuzzy decision matrix, and then we use the score function to calculate the score of each attribute value and construct the score matrix of the collective interval-valued intuitionistic fuzzy decision matrix. From the score matrix and the given attribute weight information, we establish an optimization model to determine the weights of attributes, and construct the weighted collective interval-valued intuitionistic fuzzy decision matrix, and then determine the interval-valued intuitionistic positive-ideal solution and interval-valued intuitionistic negative-ideal solution. Based on different distance definitions, we calculate the relative closeness of each alternative to the interval-valued intuitionistic positive-ideal solution and rank the alternatives according to the relative closeness to the interval-valued intuitionistic positive-ideal solution and select the most desirable one(s). Finally, an example is used to illustrate the applicability of the proposed approach.



## 1. Introduction

Decision making is the procedure to find the best alternative among a set of feasible alternatives. Multiple attribute decision making (MADM) problems (i.e., decision making problems considering several attributes) are widely spread in real life decision situation. A MADM problem can be expressed in matrix format as follows:

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| | $O_1$ | $O_2$ | $\cdots$ | $O_n$ |
|---|---|---|---|---|
| $u_1$ | $r_{11}$ | $r_{12}$ | $\cdots$ | $r_{1n}$ |
| $u_2$ | $r_{21}$ | $r_{22}$ | $\cdots$ | $r_{2n}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\cdots$ | $\vdots$ |
| $u_m$ | $r_{m1}$ | $r_{m2}$ | $\cdots$ | $r_{mn}$ |

$$w = (w_1, w_2, \ldots, w_m)^T,$$

where $O_1, O_2, \ldots, O_n$ are possible alternatives among which decision-makers have to choose, $u_1, u_2, \ldots, u_m$ are attributes with which alternative performance is measured, $r_{ij}$ is the rating of alternative $O_j$ with respective to attribute $u_i$, and $w_i$ is the weight of attribute $u_i$.

The technique for order preference by similarity to ideal solution (TOPSIS) proposed Hwang and Yoon [1] is one of well-known methods for solving classical MADM problems. The underlying logic of TOPSIS method is to define the positive-ideal solution (PIS) and the negative-ideal solution (NIS). The PIS is the solution that maximizes the benefit criteria and minimizes the cost criteria, whereas the NIS is the solution that minimizes the benefit criteria and maximizes the cost criteria. The optimal alternative is the one which the shortest distance from the positive solution and the farthest distance from the negative solution. There exists a large amount of literature involving TOPSIS theory and applications. For example, Lai et al. [2] applied the concept of TOPSIS on multiple objective decision making (MODM) problems. Abo-Sinha and Amer [3] extended TOPSIS method for solving multi-objective large-scale nonlinear programming problems. Opricovic and Tzeng [4] conducted a comparative analysis of TOPSIS and VIKOR. The VIKOR (VlseKriterijuska Optimizacija I Komoromisno Resenje) method, developed by Opricovic [5], is a compromise ranking approach. It determines a compromise solution, providing a maximum utility for the majority and a minimum regret for the opponent. There are necessary steps in utilizing TOPSIS involving numerical measures of the relative importance of attributes and the performance of each alternatives with respect to these attributes. However, exact numerical data are inadequate to model real-life situations since human judgements are often vague under many conditions. Thus, many researchers [6–12] extended TOPSIS approach to fuzzy environment as a natural generalization of TOPSIS models. For example, Jahanshaloo et al. [8] developed an algorithmic method to extend TOPSIS for decision making problems with interval data. Yang and Hung [11] utilized TOPSIS for solving a plant layout design problem. In particular, Wang and Lee [10] proposed two operator Up and Lo to find positive-ideal and negative-ideal solutions and used these operators to solve fuzzy multiple-criteria group decision making problem. Chen and Tsao [7] extended the concept of TOPSIS to develop a method for solving MADM problems with interval-valued fuzzy data and compared the results using different distance measures, including Hamming distance, Euclidean distance and their normalized forms. Sun [12] developed an evaluation model based on the fuzzy analytic hierarchy process (AHP) and fuzzy TOPSIS to help the industrial practitioners for the performance evaluations in fuzzy environment. The AHP [13] is also powerful method to solve complex decision problems. The AHP method is a multicriteria method of analysis based on an additive weighting process, in which several relevant attributes are represented through their relative importance. AHP has been extensively applied by academics and professionals [12,14–20].

In many practical decision-making problems, such as the selection of a partner for an enterprise in the field of supply chain management, military system efficiency evaluation and so on, decision-makers usually need to provide their preferences over alternatives. Consider that the socio-economic environment becomes more complex, the preference information provided by decision-makers is usually imprecise, that is, there may be hesitation or uncertainty about preferences because a decision should be made under time pressure and lack of knowledge or data, or the decision-makers have limited attention and information processing capacities. In such cases, it is suitable and convenient to express the decision-makers' preferences in interval-valued intuitionistic fuzzy sets (IVIFSs) [21,22]. The fundamental characteristic of the IVIFS is that the values of its membership function and nonmembership function are intervals rather than exact numbers. Therefore, it is necessary and interesting to pay attention to the group decision making problems with interval-valued intuitionistic preference information. Xu [23] developed some geometric aggregation operators, such as the interval-valued intuitionistic fuzzy geometric (IIFG) operator and interval-valued intuitionistic fuzzy weighted geometric (IIFWG) operator and applied them to multiple attribute group decision making (MAGDM) with interval-valued intuitionistic fuzzy information. Xu and Chen [24,25] and Wei and Wang [26], respectively, developed some geometric aggregation operators, such as the interval-valued intuitionistic fuzzy ordered weighted geometric (IIFOWG) operator and interval-valued intuitionistic fuzzy hybrid geometric (IIFHG) operator and applied them to MAGDM with interval-valued intuitionistic fuzzy information. However, they used the IIFWG, IIFWOG and IIFHG operators in the situation where the information about attribute weights is completely known. Based on the correlation coefficient [27,28] of IVIFSs, Park et al. [29] investigated the group decision making problems in which the information about attribute weights is partially known.

In this paper, we extend the concept of TOPSIS to develop a method for solving MAGDM problems in which the preference information provided by the decision-makers is presented as interval-valued intuitionistic fuzzy decision matrices where each of the elements is characterized by interval-valued intuitionistic fuzzy number (IVIFN), and the information about attribute weights is partially known. In Section 2, we briefly review the basic concepts and operations related to IVIFNs. In Section 3, we present the considered problem and use the IIFHG operator to aggregate all individual interval-valued intuitionistic fuzzy decision matrices provided by the decision-makers into the collective interval-valued intuitionistic fuzzy decision matrix, and then use the score function to calculate the score of each attribute value and construct the score matrix of the collective interval-valued intuitionistic fuzzy decision matrix. From the score matrix and the given attribute weight information, we establish an optimization model to determine the weights of attributes, and construct the weighted collective interval-valued intuitionistic fuzzy decision matrix, and then determine the interval-valued intuitionistic PIS and interval-valued intuitionistic NIS. Based on different distance definitions, we calculate the relative closeness of each alternative to the interval-valued intuitionistic PIS and rank the alternatives according to the relative closeness to the interval-valued intuitionistic PIS and select the most desirable one(s). An example to illustrate the applicability of the proposed approach is provided in Section 4 and, finally, the conclusions are discussed in Section 5.

## 2. Basic concept

Let $X$ be a non-empty and finite set with Card $(X) = n$. Let $D[0,1]$ be the set of all closed subintervals of the unit interval $[0,1]$. An interval-valued intuitionistic fuzzy set (IVIFS) [21] $A$ in $X$ is an object having the form:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}, \tag{1}$$

where $\mu_A$: $X \to D[0,1]$, $\nu_A$: $X \to D[0,1]$ with the condition $\sup \mu_A(x) + \sup\nu_A(x) \leqslant 1$ for any $x \in X$.

The intervals $\mu_A(x)$ and $\nu_A(x)$ denote, respectively, the degree of belongingness and the degree of non-belongingness of the element $x$ to $A$. Then for each $x \in X$, $\mu_A(x)$ and $\nu_A(x)$ are closed intervals and their lower and upper end points are denoted by $\mu_{AL}(x)$, $\mu_{AU}(x)$, $\nu_{AL}(x)$ and $\nu_{AU}(x)$, respectively, and thus we can replace Eq. (1) with

$$A = \{\langle x, [\mu_{AL}(x), \mu_{AU}(x)], [\nu_{AL}(x), \nu_{AU}(x)]\rangle : x \in X\}, \tag{2}$$

where $0 \leqslant \mu_{AU}(x) + \nu_{AU}(x) \leqslant 1$ for any $x \in X$.

For each IVIFS $A$ in $X$, Park et al. [30] called

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) = [1 - \mu_{AU}(x) - \nu_{AU}(x), 1 - \mu_{AL}(x) - \nu_{AL}(x)], \tag{3}$$

an intuitionistic fuzzy interval of $X$ in $A$.

For convenience, Xu [23] called $\tilde{a} = \langle [a,b],[c,d]\rangle$ an interval-valued intuitionistic fuzzy number (IVIFN), where $[a,b] \subset [0,1]$, $[c,d] \subset [0,1]$ and $b + d \leqslant 1$.

Atanassov [22] and Atanassov and Gargov [21] introduced some basic operations on IVIFSs, which not only can ensure that the operational results are IVIFSs but also are useful in the calculus of variables under interval-valued intuitionistic fuzzy environment. Motivated by the operations in [21,22], Xu [23] and Xu and Chen [24] defined three operational laws of IVIFNs, which are useful in the remainder of this paper, as follows:

Let $\tilde{a}_1 = \langle [a_1,b_1],[c_1,d_1]\rangle, \tilde{a}_2 = \langle [a_2,b_2],[c_2,d_2]\rangle$ and $\tilde{a} = \langle [a,b],[c,d]\rangle$ be three IVIFNs; then.

1) $\tilde{a}_1 \otimes \tilde{a}_2 = \langle [a_1a_2, b_1b_2], [c_1 + c_2 - c_1c_2, d_1 + d_2 - d_1d_2]\rangle$;
2) $\tilde{a}^\lambda = \langle [a^\lambda, b^\lambda], [1-(1-c)^\lambda, 1-(1-d)^\lambda]\rangle, \lambda > 0$;
3) $\lambda\tilde{a} = \langle [1-(1-a)^\lambda, 1-(1-b)^\lambda], [c^\lambda, d^\lambda]\rangle, \lambda > 0$;

which can ensure the operational results are also IVIFNs. Moreover, Xu [23] defined a score function $s$ to measure a IVIFN $\tilde{a}$ as follows:

$$s(\tilde{a}) = \frac{1}{2}(a - c + b - d), \tag{4}$$

where $s(\tilde{a}) \in [-1,1]$. The larger the value of $s(\tilde{a})$, the higher the IVIFN $\tilde{a}$. Especially, if $s(\tilde{a}) = 1$, then $\tilde{a} = \langle [1,1],[0,0]\rangle$, which is the largest IVIFN; if $s(\tilde{a}) = -1$, then $\tilde{a} = \langle [0,0],[1,1]\rangle$, which is the smallest IVIFN.

Wei and Wang [26] defined an accuracy function $h$ to evaluate the accuracy degree of a IVIFN $\tilde{a}$ as follows:

$$h(\tilde{a}) = \frac{1}{2}(a + b + c + d), \tag{5}$$

where $h(\tilde{a}) \in [0,1]$. The larger the value of $h(\tilde{a})$, the higher the accuracy degree of the IVIFN $\tilde{a}$.

From Eq. (3), we define the hesitancy degree of the IVIFN $\tilde{a} = \langle [a,b],[c,d]\rangle$ as the midpoint of intuitionistic fuzzy interval of $\tilde{a}$, i.e.,

$$\pi(\tilde{a}) = \frac{1}{2}((1 - a - c) + (1 - b - d)). \tag{6}$$

Then we get the relation between the hesitancy degree and the accuracy degree of the IVIFN $\tilde{a}$

$$\pi(\tilde{a}) = \frac{1}{2}((1 - a - c) + (1 - b - d)) = 1 - h(\tilde{a}),$$

i.e.,

$$\pi(\tilde{a}) + h(\tilde{a}) = 1. \tag{7}$$

From Eq. (7), we know that the higher the accuracy degree $h(\tilde{a})$, the lower the hesitancy degree $\pi(\tilde{a})$.

Based on the score function and the accuracy function, Xu [23] defined a method to compare two IVIFNs as follows:

**Definition 1.** Let $\tilde{a}_1 = \langle [a_1, b_1], [c_1, d_1] \rangle$ and $\tilde{a}_2 = \langle [a_2, b_2], [c_2, d_2] \rangle$ be two IVIFNs, $s(\tilde{a}_1) = \frac{1}{2}(a_1 - c_1 + b_1 - d_1)$ and $s(\tilde{a}_2) = \frac{1}{2}(a_2 - c_2 + b_2 - d_2)$ be the score of $\tilde{a}_1$ and $\tilde{a}_2$, respectively, and $h(\tilde{a}_1) = \frac{1}{2}(a_1 + b_1 + c_1 + d_1)$ and $h(\tilde{a}_2) = \frac{1}{2}(a_2 + b_2 + c_2 + d_2)$ be the accuracy degree of $\tilde{a}_1$ and $\tilde{a}_2$, respectively; then:

- if $s(\tilde{a}_1) < s(\tilde{a}_2)$, then $\tilde{a}_1$ is smaller than $\tilde{a}_2$, denoted by $\tilde{a}_1 < \tilde{a}_2$;
- if $s(\tilde{a}_1) = s(\tilde{a}_2)$, then
  (1) if $h(\tilde{a}_1) = h(\tilde{a}_2)$, then $\tilde{a}_1$ and $\tilde{a}_2$ represent the same information, i.e., $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$ and $d_1 = d_2$, denoted by $\tilde{a}_1 = \tilde{a}_2$;
  (2) if $h(\tilde{a}_1) < h(\tilde{a}_2)$, then $\tilde{a}_1$ is smaller than $\tilde{a}_2$, denoted by $\tilde{a}_1 < \tilde{a}_2$.

**Theorem 1.** *Let $\tilde{a}_1 = \langle [a_1, b_1], [c_1, d_1] \rangle$ and $\tilde{a}_2 = \langle [a_2, b_2], [c_2, d_2] \rangle$ be two IVIFs; then we have:*

$$a_1 \leqslant a_2, \quad b_1 \leqslant b_2, \quad c_1 \geqslant c_2 \quad \text{and} \quad d_1 \geqslant d_2 \Rightarrow \tilde{a}_1 \leqslant \tilde{a}_2.$$

**Proof.** Since $s(\tilde{a}_1) = \frac{1}{2}(a_1 - c_1 + b_1 - d_1), s(\tilde{a}_2) = \frac{1}{2}(a_2 - c_2 + b_2 - d_2), a_1 \leqslant a_2, b_1 \leqslant b_2, c_1 \geqslant c_2$ and $d_1 \geqslant d_2$, we have:

$$s(\tilde{a}_1) - s(\tilde{a}_2) = \frac{1}{2}(a_1 - c_1 + b_1 - d_1) - (a_2 - c_2 + b_2 - d_2) = \frac{1}{2}(a_1 - a_2) + (b_1 - b_2) + (c_2 - c_1) + (d_2 - d_1).$$

If $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$ and $d_1 = d_2$, then $\tilde{a}_1 = \tilde{a}_2$; otherwise, we have $s(\tilde{a}_1) - s(\tilde{a}_2) < 0$, i.e., $s(\tilde{a}_1) < s(\tilde{a}_2)$. Thus from Definition 1, it follows that $\tilde{a}_1 < \tilde{a}_2$, which completes the proof of Theorem 1. □

Deschrijver and Kerre [31] defined a complete lattice as a partially ordered set such that every nonempty subset of it have a supremum and an infimum, and defined a relation $\leqslant_{L^*}$ on $L^* = \{\tilde{a} = \langle [a, b], [c, d] \rangle \in D[0, 1] \times D[0, 1] : b + d \leqslant 1\}$ as follows: for any $\tilde{a}_1 = \langle [a_1, b_1], [c_1, d_1] \rangle, \tilde{a}_2 = \langle [a_2, b_2], [c_2, d_2] \rangle \in L^*$,

$$\tilde{a}_1 \leqslant_{L^*} \tilde{a}_2 \iff a_1 \leqslant a_2, \quad b_1 \leqslant b_2, \quad c_1 \geqslant c_2 \quad \text{and} \quad d_1 \geqslant d_2 \tag{8}$$

and showed that $(L^*, \leqslant_{L^*})$ is a complete lattice. However, in some situations, Eq. (8) cannot be used to compare IVIFNs. For example, let $\tilde{a}_1 = \langle [0.2, 0.4], [0.5, 0.6] \rangle$ and $\tilde{a}_2 = \langle [0.2, 0.3], [0.4, 0.7] \rangle$. Then it is impossible to know which one is bigger by using Eq. (8). But in this case, we use Definition 1 to compare them. In fact, since

$$s(\tilde{a}_1) = \frac{1}{2}(0.2 - 0.5 + 0.4 - 0.6) = -0.25, \quad s(\tilde{a}_2) = \frac{1}{2}(0.2 - 0.4 + 0.3 - 0.7) = -0.30,$$

then, by Definition 1, we know that $\tilde{a}_1 > \tilde{a}_2$.

## 3. Extended TOPSIS method for group decision making problem with interval-valued intuitionistic fuzzy data

In this section, we propose the TOPSIS method to solve MAGDM problems in which all preference information provided by decision-makers is expressed as interval-valued intuitionistic fuzzy decision matrices where each of the elements is characterized by IVIFN, and the information about attribute weights is partially known.

For MAGDM problem, let $O = \{O_1, O_2, \ldots, O_n\}$ be the set of $n$ alternatives, $D = \{d_1, d_2, \ldots, d_l\}$ be the set of $l$ decision-makers, and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)^T$ be the weight vector of decision-makers, where $\lambda_k \geqslant 0$, $k = 1, 2, \ldots, l$, and $\sum_{k=1}^{l} \lambda_k = 1$. Let $U = \{u_1, u_2, \ldots, u_m\}$ be the set of $m$ attributes. In general, the decision-makers need to determine the importance degrees of a set $U$ of $m$ attributes. Thus we suppose that the decision-makers provide the attribute weight information may be presented in the following forms [32,33], for $i \neq j$:

1. A weak ranking: $\{w_i \geqslant w_j\}$;
2. A strict ranking: $\{w_i - w_j \geqslant \delta_i (>0)\}$;
3. A ranking with multiples: $\{w_i \geqslant \delta_i w_j\}$, $0 \leqslant \delta_i \leqslant 1$;
4. An interval form: $\{\delta_i \leqslant w_i \leqslant \delta_i + \epsilon_i\}$, $0 \leqslant \delta_i \leqslant \delta_i + \epsilon_i \leqslant 1$;
5. A ranking of differences: $\{w_i - w_j \geqslant w_k - w_l\}$, for $j \neq k \neq l$.

For convenience, we denote by $H$ the set of the known information about attribute weights provided by the decision-makers. Let $R^{(k)} = \left(\tilde{r}_{ij}^{(k)}\right)_{m \times n}$ be an interval-valued intuitionistic fuzzy decision matrix, provided by decision-maker $d_k$ ($k = 1, 2, \ldots, l$), as the following form:

| | $O_1$ | $O_2$ | $\cdots$ | $O_n$ |
|---|---|---|---|---|
| $u_1$ | $\tilde{r}_{11}^{(k)}$ | $\tilde{r}_{12}^{(k)}$ | $\cdots$ | $\tilde{r}_{1n}^{(k)}$ |
| $u_2$ | $\tilde{r}_{21}^{(k)}$ | $\tilde{r}_{22}^{(k)}$ | $\cdots$ | $\tilde{r}_{2n}^{(k)}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $u_m$ | $\tilde{r}_{m1}^{(k)}$ | $\tilde{r}_{m2}^{(k)}$ | $\cdots$ | $\tilde{r}_{mn}^{(k)}$ |

where $\tilde{r}_{ij}^{(k)} = \left\langle \left[a_{ij}^{(k)}, b_{ij}^{(k)}\right], \left[c_{ij}^{(k)}, d_{ij}^{(k)}\right] \right\rangle$ is an IVIFN representing the performance rating of the alternative $O_j$ with respect to the attribute $u_i \in U$, provided by the decision-maker $d_k \in D$ (i.e., $\left[a_{ij}^{(k)}, b_{ij}^{(k)}\right]$ indicates the degree that the alternative $O_j \in O$ satisfy the attribute $u_i$, expressed by the decision-maker $d_k$, while $\left[c_{ij}^{(k)}, d_{ij}^{(k)}\right]$ indicates the degree that the alternative $O_j \in O$ does not satisfy the attribute $u_i$, expressed by the decision-maker $d_k$) and

$$\left[a_{ij}^{(k)}, b_{ij}^{(k)}\right] \subset [0,1], \left[c_{ij}^{(k)}, d_{ij}^{(k)}\right] \subset [0,1], b_{ij}^{(k)} + d_{ij}^{(k)} \leqslant 1, \quad i = 1,2,\ldots,m;\ j = 1,2,\ldots,n. \tag{9}$$

To extend TOPSIS method in the process of group decision making, we first need to fuse all individual decision opinion into group opinion. To do this, we use the IIFHG operator [24,26] to aggregate all individual interval-valued intuitionistic fuzzy decision matrices $R^{(k)} = \left(\tilde{r}_{ij}^{(k)}\right)_{m\times n} (k = 1,2,\ldots,l)$ into the collective interval-valued intuitionistic fuzzy decision matrix $R = (\tilde{r}_{ij})_{m\times n}$,

| | $O_1$ | $O_2$ | $\cdots$ | $O_n$ |
|---|---|---|---|---|
| $u_1$ | $\tilde{r}_{11}$ | $\tilde{r}_{12}$ | $\cdots$ | $\tilde{r}_{1n}$ |
| $u_2$ | $\tilde{r}_{21}$ | $\tilde{r}_{22}$ | $\cdots$ | $\tilde{r}_{2n}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $u_m$ | $\tilde{r}_{m1}$ | $\tilde{r}_{m2}$ | $\cdots$ | $\tilde{r}_{mn}$ |

where

$$\begin{aligned}\tilde{r}_{ij} &= \mathrm{IIFHG}_{\alpha,\lambda}\left(\tilde{r}_{ij}^{(1)}, \tilde{r}_{ij}^{(2)}, \ldots, \tilde{r}_{ij}^{(l)}\right) = \left(\dot{\tilde{r}}_{ij}^{(\sigma(1))}\right)^{\alpha_1} \otimes \left(\dot{\tilde{r}}_{ij}^{(\sigma(2))}\right)^{\alpha_2} \otimes \cdots \left(\dot{\tilde{r}}_{ij}^{(\sigma(l))}\right)^{\alpha_l} \\ &= \left\langle \left[\prod_{k=1}^{n} \left(\dot{a}_{ij}^{(\sigma(k))}\right)^{\alpha_k}, \prod_{k=1}^{n} \left(\dot{b}_{ij}^{(\sigma(k))}\right)^{\alpha_k}\right], \left[1 - \prod_{k=1}^{n} \left(1 - \dot{c}_{ij}^{(\sigma(k))}\right)^{\alpha_k}, 1 - \prod_{k=1}^{n} \left(1 - \dot{d}_{ij}^{(\sigma(k))}\right)^{\alpha_k}\right] \right\rangle,\end{aligned} \tag{10}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)^T$ is weight vector of IIFHG operator with $\alpha_k > 0$ $(k = 1,2,\ldots,l)$ and $\sum_{k=1}^{l} \alpha_k = 1$, and $\dot{\tilde{r}}_{ij}^{(\sigma(k))} = \left\langle \left[\dot{a}_{ij}^{\sigma((k))}, \dot{b}_{ij}^{\sigma((k))}\right], \left[\dot{c}_{ij}^{\sigma((k))}, \dot{d}_{ij}^{\sigma((k))}\right] \right\rangle$ is the $k$th largest of the weighted IVIFNs $\dot{\tilde{r}}_{ij}^{(k)} \left(\dot{\tilde{r}}_{ij}^{(k)} = \left(\tilde{r}_{ij}^{(k)}\right)^{l\lambda_k},\ i = 1,2,\ldots,m;\ j = 1,2,\ldots,n\right)$. Here, we denote by $\tilde{r}_{ij} = \langle [a_{ij}, b_{ij}], [c_{ij}, d_{ij}] \rangle,\ i = 1,2,\ldots,m;\ j = 1,2,\ldots,n$.

In the situations where the information about attribute weights is completely known, that is, the weight vector $w = (w_1, w_2, \ldots, w_m)^T$ of the attributes $u_i$ $(i = 1,2,\ldots,m)$ can be completely determined in advance, then we can construct the weighted collective interval-valued intuitionistic fuzzy decision matrix $R^* = \left(\tilde{r}_{ij}^*\right)_{m\times n}$,

| | $O_1$ | $O_2$ | $\cdots$ | $O_n$ |
|---|---|---|---|---|
| $u_1$ | $\tilde{r}_{11}^*$ | $\tilde{r}_{12}^*$ | $\cdots$ | $\tilde{r}_{1n}^*$ |
| $u_2$ | $\tilde{r}_{21}^*$ | $\tilde{r}_{22}^*$ | $\cdots$ | $\tilde{r}_{2n}^*$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $u_m$ | $\tilde{r}_{m1}^*$ | $\tilde{r}_{m2}^*$ | $\cdots$ | $\tilde{r}_{mn}^*$ |

where $\tilde{r}_{ij}^* = w_i \tilde{r}_{ij} = \left\langle [1 - (1 - a_{ij})^{w_i}, 1 - (1 - b_{ij})^{w_i}], [c_{ij}^{w_i}, d_{ij}^{w_i}] \right\rangle$ is the weighted IVIFN, $i = 1,2,\ldots,m$; $j = 1,2,\ldots,n$, and $w_i$ is weight of the attribute $u_i$ such that $w_i > 0$ and $\sum_{i=1}^{m} w_i = 1$. Now, we denote by $\tilde{r}_{ij}^* = \left\langle [a_{ij}^*, b_{ij}^*], [c_{ij}^*, d_{ij}^*] \right\rangle,\ i = 1,2,\ldots,m;$ $j = 1,2,\ldots,n$.

Let $J_1$ be a collection of benefit attributes (i.e., the larger $u_i$, the greater preference) and $J_2$ be a collection of cost attributes (i.e., the smaller $u_i$, the greater preference). The interval-valued intuitionistic PIS, denoted by $O^*$, and the interval-valued intuitionistic NIS, denoted by $O^-$, are defined as follows:

$$O^* = \left\{ \left\langle u_i, \left( \left( \max_j \tilde{r}_{ij}^* | i \in J_1 \right), \left( \min_j \tilde{r}_{ij}^* | i \in J_2 \right) \right\rangle \right| i = 1, 2, \ldots, m \right\}^T = \{\tilde{r}_1^+, \tilde{r}_2^+, \ldots, \tilde{r}_m^+\}^T, \tag{11}$$

$$O^- = \left\{ \left\langle u_i, \left( \min_j \tilde{r}_{ij}^* | i \in J_1 \right), \left( \max_j \tilde{r}_{ij}^* | i \in J_2 \right) \right\rangle \right| i = 1, 2, \ldots, m \right\}^T = \{\tilde{r}_1^-, \tilde{r}_2^-, \ldots, \tilde{r}_m^-\}^T, \tag{12}$$

where $\tilde{r}_i^+ = \langle [a_i^+, b_i^+], [c_i^+, d_i^+] \rangle$ and $\tilde{r}_i^- = \langle [a_i^-, b_i^-], [c_i^-, d_i^-] \rangle,\ i = 1, 2, \ldots, m.$

Burillo and Bustince [34] researched entropy and distance for interval-valued fuzzy sets, Grzegorzewski [35] studied distance between interval-valued fuzzy sets based on the Hausdorff metric. Park et al. [30] proposed new distance measures between interval-valued fuzzy sets and compare these measures with above-mentioned distance measures proposed by Burillo and Bustince [34] and Grzegorzewski [35], respectively. Based on these, Park et al. [30] extend three methods for measuring distances between interval-valued fuzzy sets to IVIFSs. The separation between alternatives can be measured by Hamming distance or Euclidean distance. For measuring distances between IVIFNs, we adopt several definitions proposed by Park et al. [30], including the generalizations of Hamming distance, Euclidean distance and their normalized counterparts. The separation measures, $S_{j^*}$ and $S_{j^-}$, of each alternative to the interval-valued intuitionistic PIS and interval-valued intuitionistic NIS, respectively, are derived from:

- Separation measures based on the Hamming distance
  (i) The extension of Burillo and Bustince's method, $d_1$:

$$S_{j^*}^{d_1} = \frac{1}{4} \sum_{i=1}^{m} \left[ |a_{ij}^* - a_i^+| + |b_{ij}^* - b_i^+| + |c_{ij}^* - c_i^+| + |d_{ij}^* - d_i^+| \right], \tag{13}$$

$$S_{j^-}^{d_1} = \frac{1}{4} \sum_{i=1}^{m} \left[ |a_{ij}^* - a_i^-| + |b_{ij}^* - b_i^-| + |c_{ij}^* - c_i^-| + |d_{ij}^* - d_i^-| \right]. \tag{14}$$

  (ii) The extension of modified Burillo and Bustince's method, $d_2$:

$$S_{j^*}^{d_2} = \frac{1}{4} \sum_{i=1}^{m} \left[ |a_{ij}^* - a_i^+| + |b_{ij}^* - b_i^+| + |c_{ij}^* - c_i^+| + |d_{ij}^* - d_i^+| + ||a_{ij}^* - b_{ij}^*| - |a_i^+ - b_i^+|| + ||c_{ij}^* - d_{ij}^*| - |c_i^+ - d_i^+|| \right], \tag{15}$$

$$S_{j^-}^{d_2} = \frac{1}{4} \sum_{i=1}^{m} \left[ |a_{ij}^* - a_i^-| + |b_{ij}^* - b_i^-| + |c_{ij}^* - c_i^-| + |d_{ij}^* - d_i^-| + ||a_{ij}^* - b_{ij}^*| - |a_i^- - b_i^-|| + ||c_{ij}^* - d_{ij}^*| - |c_i^- - d_i^-|| \right]. \tag{16}$$

  (iii) The extension of Grzegorzewski's method, $d_H$:

$$S_{j^*}^{d_H} = \frac{1}{2} \sum_{i=1}^{m} \left[ \max(|a_{ij}^* - a_i^+|, |b_{ij}^* - b_i^+|) + \max(|c_{ij}^* - c_i^+|, |d_{ij}^* - d_i^+|) \right], \tag{17}$$

$$S_{j^-}^{d_H} = \frac{1}{2} \sum_{i=1}^{m} \left[ \max(|a_{ij}^* - a_i^-|, |b_{ij}^* - b_i^-|) + \max(|c_{ij}^* - c_i^-|, |d_{ij}^* - d_i^-|) \right]. \tag{18}$$

- Separation measures based on the normalized Hamming distance
  (i) The extension of Burillo and Bustince's method, $l_1$:

$$S_{j^*}^{l_1} = \frac{1}{4n} \sum_{i=1}^{m} \left[ |a_{ij}^* - a_i^+| + |b_{ij}^* - b_i^+| + |c_{ij}^* - c_i^+| + |d_{ij}^* - d_i^+| \right], \tag{19}$$

$$S_{j^-}^{l_1} = \frac{1}{4n} \sum_{i=1}^{m} \left[ |a_{ij}^* - a_i^-| + |b_{ij}^* - b_i^-| + |c_{ij}^* - c_i^-| + |d_{ij}^* - d_i^-| \right]. \tag{20}$$

  (ii) The extension of modified Burillo and Bustince's method, $l_2$:

$$S_{j^*}^{l_2} = \frac{1}{4n} \sum_{i=1}^{m} \left[ |a_{ij}^* - a_i^+| + |b_{ij}^* - b_i^+| + |c_{ij}^* - c_i^+| + |d_{ij}^* - d_i^+| + ||a_{ij}^* - b_{ij}^*| - |a_i^+ - b_i^+|| + ||c_{ij}^* - d_{ij}^*| - |c_i^+ - d_i^+|| \right], \tag{21}$$

$$S_{j^-}^{l_2} = \frac{1}{4n} \sum_{i=1}^{m} \left[ |a_{ij}^* - a_i^-| + |b_{ij}^* - b_i^-| + |c_{ij}^* - c_i^-| + |d_{ij}^* - d_i^-| + ||a_{ij}^* - b_{ij}^*| - |a_i^- - b_i^-|| + ||c_{ij}^* - d_{ij}^*| - |c_i^- - d_i^-|| \right]. \tag{22}$$

(iii) The extension of Grzegorzewski's method, $l_H$:

$$S_{j^*}^{l_H} = \frac{1}{2n}\sum_{i=1}^{m}\left[\max(|a_{ij}^* - a_i^+|, |b_{ij}^* - b_i^+|) + \max(|c_{ij}^* - c_i^+|, |d_{ij}^* - d_i^+|)\right], \tag{23}$$

$$S_{j^-}^{l_H} = \frac{1}{2n}\sum_{i=1}^{m}\left[\max(|a_{ij}^* - a_i^-|, |b_{ij}^* - b_i^-|) + \max(|c_{ij}^* - c_i^-|, |d_{ij}^* - d_i^-|)\right]. \tag{24}$$

- Separation measures based on the Euclidean distance
  (i) The extension of Burillo and Bustince's method, $e_1$:

$$S_{j^*}^{e_1} = \left\{\frac{1}{4}\sum_{i=1}^{m}\left[(a_{ij}^* - a_i^+)^2 + (b_{ij}^* - b_i^+)^2 + (c_{ij}^* - c_i^+)^2 + (d_{ij}^* - d_i^+)^2\right]\right\}^{\frac{1}{2}}, \tag{25}$$

$$S_{j^-}^{e_1} = \left\{\frac{1}{4}\sum_{i=1}^{m}\left[(a_{ij}^* - a_i^-)^2 + (b_{ij}^* - b_i^-)^2 + (c_{ij}^* - c_i^-)^2 + (d_{ij}^* - d_i^-)^2\right]\right\}^{\frac{1}{2}}. \tag{26}$$

(ii) The extension of modified Burillo and Bustince's method, $e_2$:

$$S_{j^*}^{e_2} = \left\{\frac{1}{4}\sum_{i=1}^{m}\left[(a_{ij}^* - a_i^+)^2 + (b_{ij}^* - b_i^+)^2 + (c_{ij}^* - c_i^+)^2 + (d_{ij}^* - d_i^+)^2 + (|a_{ij}^* - b_{ij}^*| - |a_i^+ - b_i^+|)^2 + (|c_{ij}^* - d_{ij}^*| - |c_i^+ - d_i^+|)^2\right]\right\}^{\frac{1}{2}}, \tag{27}$$

$$S_{j^-}^{e_2} = \left\{\frac{1}{4}\sum_{i=1}^{m}\left[(a_{ij}^* - a_i^-)^2 + (b_{ij}^* - b_i^-)^2 + (c_{ij}^* - c_i^-)^2 + (d_{ij}^* - d_i^-)^2 + (|a_{ij}^* - b_{ij}^*| - |a_i^- - b_i^-|)^2 + (|c_{ij}^* - d_{ij}^*| - |c_i^- - d_i^-|)^2\right]\right\}^{\frac{1}{2}}. \tag{28}$$

(iii) The extension of Grzegorzewski's method, $e_H$:

$$S_{j^*}^{e_H} = \left\{\frac{1}{2}\sum_{i=1}^{m}\left[(\max(|a_{ij}^* - a_i^+|, |b_{ij}^* - b_i^+|))^2 + (\max(|c_{ij}^* - c_i^+|, |d_{ij}^* - d_i^+|))^2\right]\right\}^{\frac{1}{2}}, \tag{29}$$

$$S_{j^-}^{e_H} = \left\{\frac{1}{2}\sum_{i=1}^{m}\left[(\max(|a_{ij}^* - a_i^-|, |b_{ij}^* - b_i^-|))^2 + (\max(|c_{ij}^* - c_i^-|, |d_{ij}^* - d_i^-|))^2\right]\right\}^{\frac{1}{2}}. \tag{30}$$

- Separation measures based on the normalized Euclidean distance
  (i) The extension of Burillo and Bustince's method, $q_1$:

$$S_{j^*}^{q_1} = \left\{\frac{1}{4n}\sum_{i=1}^{m}\left[(a_{ij}^* - a_i^+)^2 + (b_{ij}^* - b_i^+)^2 + (c_{ij}^* - c_i^+)^2 + (d_{ij}^* - d_i^+)^2\right]\right\}^{\frac{1}{2}}, \tag{31}$$

$$S_{j^-}^{q_1} = \left\{\frac{1}{4n}\sum_{i=1}^{m}\left[(a_{ij}^* - a_i^-)^2 + (b_{ij}^* - b_i^-)^2 + (c_{ij}^* - c_i^-)^2 + (d_{ij}^* - d_i^-)^2\right]\right\}^{\frac{1}{2}}. \tag{32}$$

(ii) The extension of modified Burillo and Bustince's method, $q_2$:

$$S_{j^*}^{q_2} = \left\{\frac{1}{4n}\sum_{i=1}^{m}\left[(a_{ij}^* - a_i^+)^2 + (b_{ij}^* - b_i^+)^2 + (c_{ij}^* - c_i^+)^2 + (d_{ij}^* - d_i^+)^2 + (|a_{ij}^* - b_{ij}^*| - |a_i^+ - b_i^+|)^2 + (|c_{ij}^* - d_{ij}^*| - |c_i^+ - d_i^+|)^2\right]\right\}^{\frac{1}{2}}, \tag{33}$$

$$S_{j^-}^{q_2} = \left\{\frac{1}{4n}\sum_{i=1}^{m}\left[(a_{ij}^* - a_i^-)^2 + (b_{ij}^* - b_i^-)^2 + (c_{ij}^* - c_i^-)^2 + (d_{ij}^* - d_i^-)^2 + (|a_{ij}^* - b_{ij}^*| - |a_i^- - b_i^-|)^2 + (|c_{ij}^* - d_{ij}^*| - |c_i^- - d_i^-|)^2\right]\right\}^{\frac{1}{2}}. \tag{34}$$

(iii) The extension of Grzegorzewski's method, $q_H$:

$$S_{j^*}^{q_H} = \left\{\frac{1}{2n}\sum_{i=1}^{m}\left[(\max(|a_{ij}^* - a_i^+|, |b_{ij}^* - b_i^+|))^2 + (\max(|c_{ij}^* - c_i^+|, |d_{ij}^* - d_i^+|))^2\right]\right\}^{\frac{1}{2}}, \tag{35}$$

$$S_{j^-}^{q_H} = \left\{\frac{1}{2n}\sum_{i=1}^{m}\left[(\max(|a_{ij}^* - a_i^-|, |b_{ij}^* - b_i^-|))^2 + (\max(|c_{ij}^* - c_i^-|, |d_{ij}^* - d_i^-|))^2\right]\right\}^{\frac{1}{2}}. \tag{36}$$

The relative closeness of an alternative $O_j$ with respective to interval-valued intuitionistic PIS $O^*$ is defined as the following:

$$C_{j^*} = \frac{S_{j^-}}{S_{j^*} + S_{j^-}}, \quad j = 1, 2, \ldots, n. \tag{37}$$

The bigger the closeness coefficient $C_{j^*}$, the better the alternative $O_j$ will be, as the alternative $O_j$ is closer to the interval-valued intuitionistic PIS $O^*$. Therefore, the alternatives $O_j$ ($j = 1,2,\ldots,n$) can be ranked according to the closeness coefficients so that the best alternative can be selected.

### 3.1. A model for determining attribute weights

However, the information about attribute weights provided by the decision-makers is usually incomplete (see, [32,33]). So an interesting and important issue is how to utilize the collective interval-valued intuitionistic fuzzy decision matrix and the known weight information to find the most desirable alternative(s).

In the following, we present an approach to determining the weight of attributes.

**Definition 6.** Let $R = (\tilde{r}_{ij})_{m\times n}$ be the collective interval-valued intuitionistic fuzzy decision matrix. Then we call $S = (s_{ij})_{m\times n}$ the score matrix of $R = (\tilde{r}_{ij})_{m\times n}$, where

$$s_{ij} = s(\tilde{r}_{ij}) = \frac{1}{2}(a_{ij} - c_{ij} + b_{ij} - d_{ij}), \quad i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n \tag{38}$$

and $s(\tilde{r}_{ij})$ is called the score of $\tilde{r}_{ij}$.

Based on the score matrix, we present the overall score values of each alternatives $O_j$ ($j = 1,2,\ldots,n$):

$$s_j(w) = \sum_{i=1}^{m} w_i s_{ij}, \quad j = 1, 2, \ldots, n. \tag{39}$$

Obviously, the greater the value $s_j(w)$, the better the alternative $O_j$. When we only consider the alternative $O_j$, then a reasonable vector of attribute weights $w = (w_1,w_2,\ldots,w_m)^T$ should be determined. Thus, we establish the following optimization model to maximize $s_j(w)$:

$$(\mathbf{M}) \quad \text{Maximize} \;\; s_j(w) = \sum_{i=1}^{m} w_i s_{ij}$$

$$\text{Subject to}: \;\; w = (w_1, \ldots, w_m)^T \in H, w_i \geqslant 0, \quad i = 1, \ldots, m, \sum_{i=1}^{m} w_i = 1.$$

By solving the model (**M**), we obtain the optimal solution $w^{(j)} = \left(w_1^{(j)}, w_2^{(j)}, \ldots, w_m^{(j)}\right)^T$ corresponding to the alternative $O_j$. However, in the process of determining the weight vector $w = (w_1,w_2,\ldots,w_m)^T$, we need to consider all the alternatives $O_j$ ($j = 1,2,\ldots,n$) as a whole. Thus, we construct weight matrix $W = \left(w_i^{(j)}\right)_{m\times n}$ of the optimal solutions $w^{(j)} = \left(w_1^{(j)}, w_2^{(j)}, \ldots, w_m^{(j)}\right)^T (j = 1, 2, \ldots, n)$ as:

$$W = \begin{pmatrix} w_1^{(1)} & w_1^{(2)} & \cdots & w_1^{(n)} \\ w_2^{(1)} & w_2^{(2)} & \cdots & w_2^{(n)} \\ \vdots & \vdots & \vdots & \vdots \\ w_m^{(1)} & w_m^{(2)} & \cdots & w_m^{(n)} \end{pmatrix}$$

and we calculate the normalized eigenvector $\omega = (\omega_1,\omega_2,\ldots,\omega_n)^T$ of the matrix $(S^TW)^T(S^TW)$, and then we construct a combined weight vector as follows:

$$w = W\omega = \begin{pmatrix} w_1^{(1)} & w_1^{(2)} & \cdots & w_1^{(n)} \\ w_2^{(1)} & w_2^{(2)} & \cdots & w_2^{(n)} \\ \vdots & \vdots & \vdots & \vdots \\ w_m^{(1)} & w_m^{(2)} & \cdots & w_m^{(n)} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{pmatrix} = \omega_1 w^{(1)} + \omega_2 w^{(2)} + \cdots + \omega_n w^{(n)} \tag{40}$$

and thus we derive the weight vector $w = (w_1,w_2,\ldots,w_m)^T$ of the attributes $u_k$ ($k = 1,2,\ldots,m$).

### 3.2. An approach to MAGDM with incomplete attribute weight information

Based on the analysis above, in the following we present an approach to multiple attribute interval-valued intuitionistic fuzzy group decision making with incomplete attribute weight information:

**Step 1.** Utilize the IIFHG operator to aggregate all individual interval-valued intuitionistic fuzzy decision matrices $R^{(k)} = \left(\tilde{r}_{ij}^{(k)}\right)_{m\times n} (k = 1, 2, \ldots, l)$ into a collective interval-valued intuitionistic fuzzy decision matrix $R = (\tilde{r}_{ij})_{m\times n}$.

**Step 2.** Calculate the score matrix $S = (s_{ij})_{m\times n}$ of the collective interval-valued intuitionistic fuzzy decision matrix $R$.

**Step 3.** Utilize the model (**M**) to obtain the optimal weight vectors $w^{(j)} = \left(w_1^{(j)}, w_2^{(j)}, \ldots, w_m^{(j)}\right)^T (j = 1, 2, \ldots, n)$ corresponding to the alternatives $O_j$ $(j = 1,2,\ldots,n)$, and then construct the weight matrix $W$.

**Step 4.** Calculate the normalized eigenvector $\omega = (\omega_1,\omega_2,\ldots,\omega_n)^T$ of the matrix $(S^TW)^T(S^TW)$.

**Step 5.** Utilize Eq. (40) to derive the weight vector $w = (w_1,w_2,\ldots,w_m)^T$.

**Step 6.** Calculate the weighted collective interval-valued intuitionistic fuzzy decision matrix $R^* = (\tilde{r}_{ij}^*)_{m\times n}$.

**Step 7.** Utilize Eqs. (11) and (12) to determine the interval-valued intuitionistic PIS $O^*$ and interval-valued intuitionistic NIS $O^-$.

**Step 8.** Utilize Eqs. (13)–(36) to calculate the separation measures $S_{j^*}$ and $S_{j^-}$ of each alternative $O_j$ $(j = 1,2,\ldots,n)$ from interval-valued intuitionistic PIS $O^*$ and interval-valued intuitionistic NIS $O^-$, respectively.

**Step 9.** Utilize Eq. (37) to calculate the relative closeness $C_{j^*}$ of each alternative $O_j$ $(j = 1,2,\ldots,n)$ to the interval-valued intuitionistic PIS $O^*$.

**Step 10.** Rank the alternatives $O_j$ $(j = 1,2,\ldots,n)$, according to the relative closeness to the interval-valued intuitionistic PIS and then select the most desirable one(s).

## 4. Illustrative example

In this section, we use a multiple attribute group decision making problem of determining what kind of air-conditioning systems should be installed in a library (adapted from [36]) to illustrate the proposed approach.

A city is planning to build a municipal library. One of the problems facing the city development commissioner is to determine what kind of air-conditioning systems should be installed in the library. The contractor offers four feasible alternatives $O_j$ $(j = 1,2,3,4)$, which might be adapted to the physical structure of the library. The offered air-conditioning system must take a decision according to the following five attributes: (1) performance $(u_1)$, (2) maintainability $(u_2)$, (3) flexibility $(u_3)$, (4) cost $(u_4)$, (5) safety $(u_5)$. Let $J = \{u_1,u_2,u_3,u_4,u_5\}$ be the set of five attributes, and assume that $u_1,u_2,u_3$ and $u_5$ are benefit attributes and $u_4$ is cost attribute. That is, $J_1 = \{u_1,u_2,u_3,u_5\}$ and $J_2 = \{u_4\}$. There are a committee of four experts, $d_1$, $d_2$, $d_3$ and $d_4$, whose weight vector is $\lambda = (0.3,0.2,0.3,0.2)^T$. The experts $d_k$ $(k = 1,2,3,4)$ represent, respectively, the characteristics of the alternatives $O_j$ $(j = 1,2,3,4)$ by the IVIFNs $r_{ij}^{(k)} (i = 1, 2, 3, 4, 5, j = 1, 2, 3, 4)$ with respect to the attributes $u_i$ $(i = 1,2,3,4,5)$, listed in Tables 1–4 (i.e., interval-valued intuitionistic fuzzy decision matrices $R^{(k)} = (r_{ij}^{(k)})_{5\times 4} (k = 1, 2, 3, 4)$).

Assume that the information about attribute weights, given by decision-makers, is shown as follows, respectively:

$$d_1 : w_1 \leqslant 0.3, \quad 0.2 \leqslant w_3 \leqslant 0.5,$$

$$d_2 : 0.1 \leqslant w_2 \leqslant 0.2, \quad w_5 \leqslant 0.4,$$

$$d_3 : w_3 - w_2 \geqslant w_5 - w_4, \quad w_4 \geqslant w_1,$$

$$d_4 : w_3 - w_1 \leqslant 0.1, \quad 0.1 \leqslant w_4 \leqslant 0.3.$$

Then the set $H$ of the known information about attribute weights provided by the decision-makers is

$$H = \{w_1 \leqslant 0.3, 0.2 \leqslant w_3 \leqslant 0.5, 0.1 \leqslant w_2 \leqslant 0.2, w_5 \leqslant 0.4, w_3 - w_2 \geqslant w_5 - w_4, w_4 \geqslant w_1, w_3 - w_1 \leqslant 0.1, 0.1 \leqslant w_4 \leqslant 0.3\}.$$

**Step 1.** Utilize the IIFHG operator (let $\alpha = (0.155,0.345,0.345,0.155)^T$ be its weight vector derived by the normal distribution based method [37]) to aggregate the individual interval-valued intuitionistic fuzzy decision matrices $R^{(k)} = (r_{ij}^{(k)})_{5\times 4} (k = 1, 2, 3, 4)$ into the collective interval-valued intuitionistic fuzzy decision matrix $R = (r_{ij})_{5\times 4}$ (Table 5).

**Step 2.** Calculate the score matrix $S = (s_{ij})_{5\times 4}$ of the collective interval-valued intuitionistic fuzzy decision matrix $R$ (Table 6):

**Step 3.** Use the method (**M**) to obtain the optimal weight vectors $w^{(j)} = \left(w_1^{(j)}, w_2^{(j)}, w_3^{(j)}, w_4^{(j)}, w_5^{(j)}\right)^T (j = 1, 2, 3, 4)$ corresponding to the alternatives $O_j$ $(j = 1,2,3,4)$:

$$w^{(1)} = (0.2667, 0.1, 0.3667, 0.2667, 0)^T,$$

$$w^{(2)} = (0.16, 0.1, 0.26, 0.16, 0.32)^T,$$

$$w^{(3)} = (0.1, 0.2, 0.2, 0.25, 0.25)^T,$$

$$w^{(4)} = (0.3, 0.1, 0.2, 0.3, 0.1)^T$$

**Table 1**
Interval-valued intuitionistic fuzzy decision matrix $R^{(1)}$.

| | $O_1$ | $O_2$ | $O_3$ | $O_4$ |
|---|---|---|---|---|
| $u_1$ | $\langle[0.5,0.6],[0.2,0.3]\rangle$ | $\langle[0.3,0.4],[0.4,0.6]\rangle$ | $\langle[0.4,0.5],[0.3,0.5]\rangle$ | $\langle[0.3,0.5],[0.4,0.5]\rangle$ |
| $u_2$ | $\langle[0.3,0.5],[0.4,0.5]\rangle$ | $\langle[0.1,0.3],[0.2,0.4]\rangle$ | $\langle[0.7,0.8],[0.1,0.2]\rangle$ | $\langle[0.1,0.2],[0.7,0.8]\rangle$ |
| $u_3$ | $\langle[0.6,0.7],[0.2,0.3]\rangle$ | $\langle[0.3,0.4],[0.4,0.5]\rangle$ | $\langle[0.5,0.8],[0.1,0.2]\rangle$ | $\langle[0.1,0.2],[0.5,0.8]\rangle$ |
| $u_4$ | $\langle[0.5,0.7],[0.1,0.2]\rangle$ | $\langle[0.2,0.4],[0.5,0.6]\rangle$ | $\langle[0.4,0.6],[0.2,0.3]\rangle$ | $\langle[0.2,0.3],[0.4,0.6]\rangle$ |
| $u_5$ | $\langle[0.1,0.4],[0.3,0.5]\rangle$ | $\langle[0.7,0.8],[0.1,0.2]\rangle$ | $\langle[0.5,0.6],[0.2,0.3]\rangle$ | $\langle[0.2,0.3],[0.5,0.6]\rangle$ |

**Table 2**
Interval-valued intuitionistic fuzzy decision matrix $R^{(2)}$.

| | $O_1$ | $O_2$ | $O_3$ | $O_4$ |
|---|---|---|---|---|
| $u_1$ | $\langle[0.4,0.5],[0.2,0.4]\rangle$ | $\langle[0.3,0.5],[0.4,0.5]\rangle$ | $\langle[0.4,0.6],[0.3,0.4]\rangle$ | $\langle[0.3,0.4],[0.4,0.6]\rangle$ |
| $u_2$ | $\langle[0.3,0.4],[0.4,0.6]\rangle$ | $\langle[0.1,0.3],[0.3,0.7]\rangle$ | $\langle[0.6,0.8],[0.1,0.2]\rangle$ | $\langle[0.1,0.2],[0.6,0.8]\rangle$ |
| $u_3$ | $\langle[0.6,0.7],[0.1,0.2]\rangle$ | $\langle[0.3,0.4],[0.4,0.5]\rangle$ | $\langle[0.7,0.8],[0.1,0.2]\rangle$ | $\langle[0.1,0.2],[0.7,0.8]\rangle$ |
| $u_4$ | $\langle[0.5,0.6],[0.1,0.3]\rangle$ | $\langle[0.2,0.3],[0.6,0.7]\rangle$ | $\langle[0.4,0.6],[0.3,0.4]\rangle$ | $\langle[0.3,0.4],[0.4,0.6]\rangle$ |
| $u_5$ | $\langle[0.1,0.3],[0.3,0.5]\rangle$ | $\langle[0.6,0.8],[0.1,0.2]\rangle$ | $\langle[0.5,0.6],[0.2,0.4]\rangle$ | $\langle[0.2,0.4],[0.5,0.6]\rangle$ |

**Table 3**
Interval-valued intuitionistic fuzzy decision matrix $R^{(3)}$.

| | $O_1$ | $O_2$ | $O_3$ | $O_4$ |
|---|---|---|---|---|
| $u_1$ | $\langle[0.4,0.7],[0.1,0.2]\rangle$ | $\langle[0.4,0.5],[0.2,0.4]\rangle$ | $\langle[0.2,0.4],[0.3,0.4]\rangle$ | $\langle[0.3,0.4],[0.2,0.4]\rangle$ |
| $u_2$ | $\langle[0.3,0.5],[0.3,0.4]\rangle$ | $\langle[0.2,0.4],[0.4,0.5]\rangle$ | $\langle[0.6,0.8],[0.1,0.2]\rangle$ | $\langle[0.1,0.2],[0.6,0.8]\rangle$ |
| $u_3$ | $\langle[0.6,0.7],[0.1,0.2]\rangle$ | $\langle[0.4,0.5],[0.3,0.4]\rangle$ | $\langle[0.5,0.7],[0.1,0.3]\rangle$ | $\langle[0.1,0.3],[0.5,0.7]\rangle$ |
| $u_4$ | $\langle[0.5,0.6],[0.1,0.3]\rangle$ | $\langle[0.1,0.2],[0.7,0.8]\rangle$ | $\langle[0.5,0.7],[0.2,0.3]\rangle$ | $\langle[0.2,0.3],[0.5,0.7]\rangle$ |
| $u_5$ | $\langle[0.3,0.5],[0.4,0.5]\rangle$ | $\langle[0.6,0.7],[0.2,0.3]\rangle$ | $\langle[0.6,0.8],[0.1,0.2]\rangle$ | $\langle[0.1,0.2],[0.6,0.8]\rangle$ |

**Table 4**
Interval-valued intuitionistic fuzzy decision matrix $R^{(4)}$.

| | $O_1$ | $O_2$ | $O_3$ | $O_4$ |
|---|---|---|---|---|
| $u_1$ | $\langle[0.6,0.7],[0.2,0.3]\rangle$ | $\langle[0.4,0.5],[0.4,0.5]\rangle$ | $\langle[0.4,0.5],[0.3,0.4]\rangle$ | $\langle[0.3,0.4],[0.4,0.5]\rangle$ |
| $u_2$ | $\langle[0.3,0.4],[0.3,0.4]\rangle$ | $\langle[0.1,0.2],[0.2,0.3]\rangle$ | $\langle[0.6,0.7],[0.1,0.3]\rangle$ | $\langle[0.1,0.3],[0.6,0.7]\rangle$ |
| $u_3$ | $\langle[0.7,0.8],[0.1,0.2]\rangle$ | $\langle[0.3,0.4],[0.5,0.6]\rangle$ | $\langle[0.5,0.8],[0.1,0.2]\rangle$ | $\langle[0.1,0.2],[0.5,0.8]\rangle$ |
| $u_4$ | $\langle[0.5,0.6],[0.1,0.3]\rangle$ | $\langle[0.2,0.3],[0.4,0.6]\rangle$ | $\langle[0.4,0.5],[0.2,0.3]\rangle$ | $\langle[0.2,0.3],[0.4,0.5]\rangle$ |
| $u_5$ | $\langle[0.1,0.2],[0.5,0.7]\rangle$ | $\langle[0.6,0.7],[0.1,0.2]\rangle$ | $\langle[0.5,0.6],[0.3,0.4]\rangle$ | $\langle[0.3,0.4],[0.5,0.6]\rangle$ |

**Table 5**
Collective interval-valued intuitionistic fuzzy decision matrix $R$.

| | $O_1$ | $O_2$ | $O_3$ | $O_4$ |
|---|---|---|---|---|
| $u_1$ | $\langle[0.4385,0.6199],[0.1549,0.2848]\rangle$ | $\langle[0.3502,0.4797],[0.3114,0.4681]\rangle$ | $\langle[0.3516,0.4906],[0.2940,0.4214]\rangle$ | $\langle[0.3000,0.4170],[0.3114,0.4887]\rangle$ |
| $u_2$ | $\langle[0.3000,0.4573],[0.3404,0.4710]\rangle$ | $\langle[0.1138,0.3010],[0.2511,0.4773]\rangle$ | $\langle[0.6395,0.7711],[0.0980,0.2263]\rangle$ | $\langle[0.1000,0.2103],[0.6012,0.7678]\rangle$ |
| $u_3$ | $\langle[0.6116,0.7117],[0.1089,0.2083]\rangle$ | $\langle[0.3379,0.4387],[0.3872,0.4887]\rangle$ | $\langle[0.5213,0.7804],[0.0980,0.2083]\rangle$ | $\langle[0.1000,0.2366],[0.5577,0.7569]\rangle$ |
| $u_4$ | $\langle[0.5000,0.6395],[0.0980,0.2567]\rangle$ | $\langle[0.1758,0.3134],[0.5305,0.6496]\rangle$ | $\langle[0.4387,0.6252],[0.2263,0.3262]\rangle$ | $\langle[0.2103,0.3109],[0.4050,0.5613]\rangle$ |
| $u_5$ | $\langle[0.1323,0.3623],[0.3747,0.5482]\rangle$ | $\langle[0.6395,0.7521],[0.1089,0.2083]\rangle$ | $\langle[0.5452,0.6502],[0.1770,0.3005]\rangle$ | $\langle[0.1849,0.3121],[0.5031,0.6118]\rangle$ |

**Table 6**
Collective score matrix $S$.

| | $O_1$ | $O_2$ | $O_3$ | $O_4$ |
|---|---|---|---|---|
| $u_1$ | 0.3093 | 0.0252 | 0.0634 | −0.0415 |
| $u_2$ | −0.0270 | −0.1568 | 0.5431 | −0.5294 |
| $u_3$ | 0.5030 | −0.0496 | 0.4977 | −0.4890 |
| $u_4$ | 0.3924 | −0.3454 | 0.2557 | −0.2225 |
| $u_5$ | −0.2141 | 0.5372 | 0.3589 | −0.3089 |

**Table 7**
Weighted collective interval-valued intuitionistic fuzzy decision matrix $R^*$.

| | $O_1$ | $O_2$ | $O_3$ | $O_4$ |
|---|---|---|---|---|
| $u_1$ | $\langle[0.1125,0.1814],[0.6798,0.7711]\rangle$ | $\langle[0.0853,0.1264],[0.7855,0.8547]\rangle$ | $\langle[0.0857,0.1303],[0.7762,0.8363]\rangle$ | $\langle[0.0711,0.1056],[0.7855,0.8623]\rangle$ |
| $u_2$ | $\langle[0.0437,0.0737],[0.8738,0.9100]\rangle$ | $\langle[0.0150,0.0438],[0.8411,0.9116]\rangle$ | $\langle[0.1199,0.1685],[0.7476,0.8303]\rangle$ | $\langle[0.0131,0.0291],[0.9383,0.9675]\rangle$ |
| $u_3$ | $\langle[0.2183,0.2766],[0.5614,0.6646]\rangle$ | $\langle[0.1018,0.1396],[0.7811,0.8299]\rangle$ | $\langle[0.1746,0.3261],[0.5461,0.6646]\rangle$ | $\langle[0.0271,0.0679],[0.8589,0.9300]\rangle$ |
| $u_4$ | $\langle[0.1560,0.2210],[0.5664,0.7169]\rangle$ | $\langle[0.0462,0.0879],[0.8563,0.8998]\rangle$ | $\langle[0.1318,0.2135],[0.6952,0.7602]\rangle$ | $\langle[0.0561,0.0871],[0.8016,0.8682]\rangle$ |
| $u_5$ | $\langle[0.0228,0.0706],[0.8524,0.9068]\rangle$ | $\langle[0.1530,0.2030],[0.6972,0.7747]\rangle$ | $\langle[0.1203,0.1571],[0.7545,0.8223]\rangle$ | $\langle[0.0327,0.0591],[0.8942,0.9232]\rangle$ |

and construct the weight matrix

$$W=\begin{pmatrix} 0.2667 & 0.16 & 0.1 & 0.3\\ 0.1 & 0.1 & 0.2 & 0.1\\ 0.3667 & 0.26 & 0.2 & 0.2\\ 0.2667 & 0.16 & 0.25 & 0.3\\ 0 & 0.32 & 0.25 & 0.1 \end{pmatrix},$$

then

$$(S^TW)^T(S^TW)=\begin{pmatrix} 0.3455 & 0.2621 & 0.2835 & 0.2848\\ 0.2621 & 0.2634 & 0.2683 & 0.2266\\ 0.2835 & 0.2683 & 0.2808 & 0.2423\\ 0.2848 & 0.2266 & 0.2423 & 0.2365 \end{pmatrix}.$$

**Step 4.** Calculate the normalized eigenvectors $\omega$ of the matrix $(S^TW)^T(S^TW)$:

$$\omega=(0.2764,0.2390,0.2519,0.2326)^T.$$

**Step 5.** Use Eq. (40) to derive the weight vector $w$:

$$w=W\omega=\begin{pmatrix} 0.2667 & 0.16 & 0.1 & 0.3\\ 0.1 & 0.1 & 0.2 & 0.1\\ 0.3667 & 0.26 & 0.2 & 0.2\\ 0.2667 & 0.16 & 0.25 & 0.3\\ 0 & 0.32 & 0.25 & 0.1 \end{pmatrix}\begin{pmatrix} 0.2764\\ 0.2390\\ 0.2519\\ 0.2326 \end{pmatrix}=(0.2069,0.1252,0.2604,0.2447,0.1627)^T.$$

**Table 8**
Separation measures for the example.

| | $S_{j^*}^{d_1}$ | $S_{j^-}^{d_1}$ | $S_{j^*}^{d_2}$ | $S_{j^-}^{d_2}$ | $S_{j^*}^{d_H}$ | $S_{j^-}^{d_H}$ |
|---|---|---|---|---|---|---|
| *Separation measures based on Hamming distance* | | | | | | |
| $O_1$ | 0.4385 | 0.3877 | 0.5201 | 0.4300 | 0.4982 | 0.4251 |
| $O_2$ | 0.3339 | 0.4655 | 0.4002 | 0.5395 | 0.4002 | 0.5395 |
| $O_3$ | 0.2337 | 0.5657 | 0.2687 | 0.6713 | 0.2687 | 0.6713 |
| $O_4$ | 0.6447 | 0.1551 | 0.7445 | 0.1846 | 0.7441 | 0.1846 |
| | $S_{j^*}^{l_1}$ | $S_{j^-}^{l_1}$ | $S_{j^*}^{l_2}$ | $S_{j^-}^{l_2}$ | $S_{j^*}^{l_H}$ | $S_{j^-}^{l_H}$ |
| *Separation measures based on normalized Hamming distance* | | | | | | |
| $O_1$ | 0.0877 | 0.0775 | 0.1040 | 0.0860 | 0.0996 | 0.0850 |
| $O_2$ | 0.0668 | 0.0931 | 0.0800 | 0.1079 | 0.0800 | 0.1079 |
| $O_3$ | 0.0467 | 0.1131 | 0.0537 | 0.1343 | 0.0537 | 0.1343 |
| $O_4$ | 0.1289 | 0.0310 | 0.1489 | 0.0369 | 0.1488 | 0.0369 |
| | $S_{j^*}^{e_1}$ | $S_{j^-}^{e_1}$ | $S_{j^*}^{e_2}$ | $S_{j^-}^{e_2}$ | $S_{j^*}^{e_H}$ | $S_{j^-}^{e_H}$ |
| *Separation measures based on Euclidean distance* | | | | | | |
| $O_1$ | 0.2578 | 0.2639 | 0.2691 | 0.2659 | 0.2924 | 0.2802 |
| $O_2$ | 0.2159 | 0.2663 | 0.2270 | 0.2743 | 0.2535 | 0.3058 |
| $O_3$ | 0.1534 | 0.3203 | 0.1566 | 0.3311 | 0.1718 | 0.3660 |
| $O_4$ | 0.3427 | 0.1629 | 0.3511 | 0.1691 | 0.3872 | 0.1914 |
| | $S_{j^*}^{q_1}$ | $S_{j^-}^{q_1}$ | $S_{j^*}^{q_2}$ | $S_{j^-}^{q_2}$ | $S_{j^*}^{q_H}$ | $S_{j^-}^{q_H}$ |
| *Separation measures based on normalized Euclidean distance* | | | | | | |
| $O_1$ | 0.1153 | 0.1180 | 0.1203 | 0.1189 | 0.1308 | 0.1253 |
| $O_2$ | 0.0966 | 0.1191 | 0.1015 | 0.1227 | 0.1134 | 0.1368 |
| $O_3$ | 0.0686 | 0.1431 | 0.0700 | 0.1481 | 0.0768 | 0.1637 |
| $O_4$ | 0.1533 | 0.0728 | 0.1570 | 0.0756 | 0.1732 | 0.0856 |

**Table 9**
The relative closeness of each alternative for the example.

| | Value | Rank | Value | Rank | Value | Rank |
|---|---|---|---|---|---|---|
| | $C_j^{d_1}$ | | $C_j^{d_2}$ | | $C_j^{d_H}$ | |
| $O_1$ | 0.4692 | 3 | 0.4526 | 3 | 0.4604 | 3 |
| $O_2$ | 0.5823 | 2 | 0.5741 | 2 | 0.5741 | 2 |
| $O_3$ | 0.7076 | 1 | 0.7141 | 1 | 0.7141 | 1 |
| $O_4$ | 0.1939 | 4 | 0.1987 | 4 | 0.1987 | 4 |
| | $C_j^{l_1}$ | | $C_j^{l_2}$ | | $C_j^{l_H}$ | |
| $O_1$ | 0.4692 | 3 | 0.4526 | 3 | 0.4604 | 3 |
| $O_2$ | 0.5823 | 2 | 0.5741 | 2 | 0.5741 | 2 |
| $O_3$ | 0.7076 | 1 | 0.7141 | 1 | 0.7141 | 1 |
| $O_4$ | 0.1939 | 4 | 0.1987 | 4 | 0.1987 | 4 |
| | $C_j^{e_1}$ | | $C_j^{e_2}$ | | $C_j^{e_H}$ | |
| $O_1$ | 0.5059 | 3 | 0.4970 | 3 | 0.4894 | 3 |
| $O_2$ | 0.5522 | 2 | 0.5472 | 2 | 0.5468 | 2 |
| $O_3$ | 0.6762 | 1 | 0.6789 | 1 | 0.6806 | 1 |
| $O_4$ | 0.3222 | 4 | 0.3250 | 4 | 0.3308 | 4 |
| | $C_j^{q_1}$ | | $C_j^{q_2}$ | | $C_j^{q_H}$ | |
| $O_1$ | 0.5059 | 3 | 0.4970 | 3 | 0.4894 | 3 |
| $O_2$ | 0.5522 | 2 | 0.5472 | 2 | 0.5468 | 2 |
| $O_3$ | 0.6762 | 1 | 0.6789 | 1 | 0.6806 | 1 |
| $O_4$ | 0.3222 | 4 | 0.3250 | 4 | 0.3308 | 4 |

**Step 6.** Calculate the weighted collective interval-valued intuitionistic fuzzy decision matrix $R^* = (\tilde{r}^*_{ij})_{m\times n}$ (Table 7):

**Step 7.** Utilize Eqs. (11) and (12) to determine the interval-valued intuitionistic PIS $O^*$ and interval-valued intuitionistic NIS $O^-$:

$$O^* = \{\langle[0.1125, 0.1814], [0.6798, 0.7711]\rangle, \langle[0.1199, 0.1685], [0.7476, 0.8303]\rangle, \langle[0.1746, 0.3261], [0.5461, 0.6646]\rangle, \langle[0.0462, 0.0879], [0.8563, 0.8998]\rangle, \langle[0.1530, 0.2030], [0.6972, 0.7747]\rangle\}^T$$

and

$$O^- = \{\langle[0.0711, 0.1056], [0.7855, 0.8623]\rangle, \langle[0.0131, 0.0291], [0.9383, 0.9675]\rangle, \langle[0.0271, 0.0679], [0.8589, 0.9300]\rangle, \langle[0.1560, 0.2210], [0.5664, 0.7169]\rangle, \langle[0.0327, 0.0591], [0.8942, 0.9232]\rangle\}^T.$$

**Step 8.** Utilize Eqs. (13)–(36) to calculate the separation measures $S_{j^*}$ and $S_{j^-}$ of each alternative $O_j$ ($j$ = 1,2,3,4) from interval-valued intuitionistic PIS $O^*$ and interval-valued intuitionistic NIS $O^-$, respectively, based on the Hamming distance, the Euclidean distance and the normalized versions (Table 8).

**Step 9.** Utilize Eq. (37) to calculate the relative closeness $C_{j^*}$ of each alternative $O_j$ ($j$ = 1,2,3,4) to the interval-valued intuitionistic PIS $O^*$ with the different separation measures, including $C_j^{d_1}, C_j^{d_2}, C_j^{d_H}$ based on the Hamming distance, $C_j^{l_1}, C_j^{l_2}, C_j^{l_H}$ based on the normalized Hamming distance, $C_j^{e_1}, C_j^{e_2}, C_j^{e_H}$ based on the Euclidean distance, and $C_j^{q_1}, C_j^{q_2}, C_j^{q_H}$ based on the normalized Euclidean distance (Table 9).

**Step 10.** Rank the preference order of alternatives $O_j$ ($j$ = 1,2,3,4) (Table 9), according to the relative closeness to the interval-valued intuitionistic PIS $O^*$ and then the most desirable alternative is $O_3$.

**Remark.** The relative closeness and corresponding preference order based on Hamming distance are the same as the results based on the normalized counterpart. The rule also holds in the cases of the Euclidean distance and its normalized version. That is,

$$C_{j^*}^{l_k} = \frac{S_{j^-}^{l_k}}{S_{j^*}^{l_k} + S_{j^-}^{l_k}} = \frac{\frac{1}{n}S_{j^-}^{d_k}}{\frac{1}{n}(S_{j^*}^{d_k} + S_{j^-}^{d_k})} = C_{j^*}^{d_k} \quad \text{for all } k = 1, 2, H,$$

$$C_{j^*}^{q_k} = \frac{S_{j^-}^{q_k}}{S_{j^*}^{q_k} + S_{j^-}^{q_k}} = \frac{\frac{1}{\sqrt{n}}S_{j^-}^{e_k}}{\frac{1}{\sqrt{n}}(S_{j^*}^{e_k} + S_{j^-}^{e_k})} = C_{j^*}^{e_k} \quad \text{for all } k = 1, 2, H.$$

We obtain six results from 12 distance measures: (i) $C_j^{d_1} = C_j^{l_1}$; (ii) $C_j^{d_2} = C_j^{l_2}$; (iii) $C_j^{d_H} = C_j^{l_H}$; (iv) $C_j^{e_1} = C_j^{q_1}$; (v) $C_j^{e_2} = C_j^{q_2}$; and (vi) $C_j^{e_H} = C_j^{q_H}$.

## 5. Conclusions

We investigate the MAGDM problems under interval-valued intuitionistic fuzzy environment, and extend TOPSIS method to handling the situations where the attribute values are characterized by IVIFNs, and the information about attribute weights is partially known. The proposed approach first fuses all individual interval-valued intuitionistic fuzzy decision matrices into the collective interval-valued intuitionistic fuzzy decision matrix by using the IIFHG operator. Next, in the situation where the information about attribute weights is incomplete, we construct the score matrix of the collective interval-valued intuitionistic fuzzy decision matrix, and established an optimization model to determine the attribute weights. Then we construct the weighted collective interval-valued intuitionistic fuzzy decision matrix and determine the interval-valued intuitionistic PIS and interval-valued intuitionistic NIS. Based on different distance definitions, we calculate the relative closeness of each alternative to the interval-valued intuitionistic PIS and rank the alternatives according to the relative closeness to the interval-valued intuitionistic PIS and select the most desirable one(s). The proposed approach in this paper not only can comfort the influence of unjust arguments on the decision results, but also avoid losing or distorting the original decision information in the process of aggregation. Thus, the proposed approach provides us a effective and practical way to deal with MAGDM problems, where the attribute values are characterized by IVIFNs and the information about attribute weights is partially known.

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