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Extension of the VIKOR method to dynamic intuitionistic fuzzy multiple attribute decision making[☆]

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ABSTRACT

In this paper, we extend the VIKOR method for dynamic intuitionistic fuzzy multiple attribute decision making (DIF-MADM). Two new aggregation operators called dynamic intuitionistic fuzzy weighted geometric (DIFWG) operator and uncertain dynamic intuitionistic fuzzy weighted geometric (UDIFWG) operator are presented. Based on the DIFWA and UDIFWA operators respectively, we develop two procedures to solve the DIF-MADM problems where all attribute values are expressed in intuitionistic fuzzy numbers or interval-valued intuitionistic fuzzy numbers, which are collected at different periods. Finally, a numerical example is used to illustrate the applicability of the proposed approach.

1. Introduction

As extension of Zadeh's fuzzy set [1] whose basic component is only a membership function, Atanassov [2–4] introduced the concept of intuitionistic fuzzy sets (IFSs). Bustince and Burillo [5] showed that IFS are vague sets [6]. IFSs has been proven to be highly useful to deal with uncertainty and vagueness, and a lot of work has been done to develop and enrich the IFS theory [7,8]. In many complex decision making problems, the decision information provided by the decision maker is often imprecise or uncertain [9] due to time pressure, lack of data, or the decision maker's limited attention and information processing capabilities. Thus, IFS is a very suitable tool to be used to describe imprecise or uncertain decision information. Recently, some researchers have shown great interest in IFS theory and applied it to the field of decision making [10–18]. Li [19] extended the linear programming techniques for multi-dimensional analysis of preference (LINMAP) to develop a new methodology for solving multi-attribute decision making problems under intuitionistic fuzzy environments. Xu and Yager [20] developed some geometric aggregation operators, which extend the traditional weighted geometric operator and ordered weighed geometric operator to accommodate the environment where the given arguments are IFSs, and developed an approach, based on the intuitionistic fuzzy hybrid geometric operator, to multi-attribute decision making based on IFSs. Liu and Wang [21] gave an evaluation function for the decision making problems to measure the degrees to which alternatives satisfy and do not satisfy the decision maker's requirement. Then, they introduced the intuitionistic fuzzy point operators, and defined a series of new score functions for multi-attribute decision making problems based on intuitionistic fuzzy point operators and evaluation function. Li et al. [22] developed a new methodology for solving multiple attribute

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group decision making problems using IFSs, in which multiple attributes are explicitly considered. In this methodology, for each decision maker in group two auxiliary fractional programming models are derived from the TOPSIS method to determine the relative closeness coefficient intervals of alternatives, which are aggregated for the group to generate the ranking order of all alternatives by computing their optimal degrees of membership based on the ranking method of interval numbers. All these studies are focused on decision making problems where all original decision information is provided in the same period. However, in many decision areas, such as multi-period investment decision making, medical diagnosis, personnel dynamic examination, and military system efficiency dynamic evaluation, the original decision information is usually collected at different periods. Thus, it is necessary to develop some approaches to deal with these issues. Xu and Yager [23] developed two new aggregation operators such as the dynamic intuitionistic fuzzy weighted averaging (DIFWA) operator and uncertain dynamic intuitionistic fuzzy weighted averaging (UDIFWA) operator, and developed two procedures, based on DIFWA and UDIFWA operators respectively, to solve the dynamic intuitionistic fuzzy multi-attribute decision making (DIF-MADM) problems where all attribute values are expressed in intuitionistic fuzzy numbers or interval-valued intuitionistic fuzzy numbers, which are collected at different periods.

The VIKOR method proposed by Opricovic [24], was developed for multiple attribute optimizations of complex systems. The VIKOR method is a compromise ranking approach for multiple criteria decision making problems. It determines a compromise solution, providing a maximum utility for the majority and a minimum regret for the opponent. There exists a large amount of literature involving VIKOR theory and application. For example, Opricovic and Tzeng [25] suggested using fuzzy logic for the VIKOR method. Tzeng et al. [26] used and compared the VIKOR and TOPSIS methods in solving a public transportation problem. Büyüközkan and Ruan [27] extended the VIKOR method to effectively solve software evaluation problem under a fuzzy environment. Opricovic and Tzeng [28] extended the VIKOR method with a stability analysis determining the weight stability intervals and with trade-offs analysis and compared the extended VIKOR method with three multicriteria decision making methods: TOPSIS, PROMETHEE, and ELECTRE. Sayadi et al. [29] extended the VIKOR method to multiple attribute decision making problem with interval numbers. Chang and Hsu [30] showed that the VIKOR method is advantageous for evaluating the relative environmental vulnerability of subdivisions in a watershed. According to a comparative analysis of VIKOR and TOPSIS written by Opricovic and Tzeng [31], the VIKOR and TOPSIS methods, respectively, use different aggregation functions and different normalization methods. The TOPSIS method is suitable for cautious decision maker(s), because the decision maker(s) might like to have a decision which not only makes as much profit as possible, but also avoids as much risk as possible, whereas the VIKOR method is suitable for those situations in which the decision maker wants to have maximum profit and the risk of decisions is less important for him.

In this paper, we shall extend the VIKOR method to solve the DIF-MADM problems. To do that, we first develop an aggregation operator called dynamic intuitionistic fuzzy weighted geometric (DIFWG) operator, and then develop a procedure for DIF-MADM. Furthermore, we extend the develop operator and procedure to deal with the situation where all the attribute values are expressed in interval-valued intuitionistic fuzzy numbers collected at different periods. Finally, a numerical example is used to illustrate the applicability of the proposed approach.

2. Preliminaries

Let us first define some basic concepts related to IFSs [2].

Definition 1. Let X be a fixed set, a fuzzy set F in X is given by Zadeh [1] as follows:

$$F = \{\langle x, \mu_F(x)\rangle | x \in X\}, \tag{1}$$

where $\mu_F : X \to [0, 1]$ denotes the membership function of the set F.

Definition 2. Let X be a fixed set, an IFS A in X is given by Atanassov [2] as an object having the following form:

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}, \tag{2}$$

where $\mu_A : X \to [0, 1]$ and $\nu_A : X \to [0, 1]$ denote, respectively, membership function and non-membership function of A with the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for any $x \in X$.

For each IFS A in X,

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \tag{3}$$

is called the degree of indeterminacy of x to A, or called the degree of hesitancy of x to A. Especially, if $\pi_A(x) = 0$ for all $x \in X$, then the IFS is reduced to a fuzzy set.

Clearly, a prominent characteristic of an IFS is that it assigns to each element a membership degree, a non-membership degree and a hesitation degree, and thus, IFS constitutes an extension of Zadeh's fuzzy set which only assigns to each element a membership degree.

For convenience of computation, Xu and Yager [20] called $\alpha = (\mu_\alpha, \nu_\alpha, \pi_\alpha)$ an intuitionistic fuzzy number[2] (IFN), where

$$\mu_\alpha \in [0, 1],\ \nu_\alpha \in [0, 1], \quad \mu_\alpha + \nu_\alpha \le 1, \qquad \pi_\alpha = 1 - \mu_\alpha - \nu_\alpha. \tag{4}$$

For an IFN $\alpha = (\mu_\alpha, \nu_\alpha, \pi_\alpha)$, if the value μ_α gets bigger and the value ν_α gets smaller, then the IFN α gets greater, and thus from (4), we know that $(1, 0, 0)$ and $(0, 1, 0)$ are the largest and smallest IFNs, respectively.

Based on the score function and the accuracy function, we define a method to compare two IFNs as follows:

Definition 3. Let $\alpha_1 = (\mu_{\alpha_1}, \nu_{\alpha_1}, \pi_{\alpha_1})$ and $\alpha_2 = (\mu_{\alpha_2}, \nu_{\alpha_2}, \pi_{\alpha_2})$ be two IFNs, $s(\alpha_1) = \mu_{\alpha_1} - \nu_{\alpha_1}$ and $s(\alpha_2) = \mu_{\alpha_2} - \nu_{\alpha_2}$ be the score of α_1 and α_2, respectively, and $h(\alpha_1) = \mu_{\alpha_1} + \nu_{\alpha_1}$ and $h(\alpha_2) = \mu_{\alpha_2} + \nu_{\alpha_2}$ be the accuracy degree of $\tilde{a}_1$ and $\tilde{a}_2$, respectively, then:

- if $s(\alpha_1) < s(\alpha_2)$, then α_1 is smaller than α_2, denoted by $\alpha_1 < \alpha_2$;
- if $s(\alpha_1) = s(\alpha_2)$, then
  (1) if $h(\alpha_1) = h(\alpha_2)$, then α_1 and α_2 represent the same information, i.e., $\mu_{\alpha_1} = \mu_{\alpha_2}$, $\nu_{\alpha_1} = \nu_{\alpha_2}$, and $\pi_{\alpha_1} = \pi_{\alpha_2}$, denoted by $\alpha_1 = \alpha_2$;
  (2) if $h(\alpha_1) < h(\alpha_2)$, then α_1 is smaller than α_2, denoted by $\alpha_1 < \alpha_2$.

Three methods of defining the Hamming distance between IFSs have been proposed by Burillo and Bustince [34], Szmidt and Kacprzyk [35], and Grzegorzewski [36], respectively. So we adopt these methods to define Hamming distance between IFNs as follows:

Definition 4. Let $\alpha_1 = (\mu_{\alpha_1}, \nu_{\alpha_1}, \pi_{\alpha_1})$ and $\alpha_2 = (\mu_{\alpha_2}, \nu_{\alpha_2}, \pi_{\alpha_2})$ be IFNs, then

- Hamming distance proposed by Burillo and Bustince, d_1:

$$d_1(\alpha_1, \alpha_2) = \frac{1}{2}(|\mu_{\alpha_1} - \mu_{\alpha_2}| + |\nu_{\alpha_1} - \nu_{\alpha_2}|); \tag{5}$$

- Hamming distance proposed by Szmidt and Kacprzyk, d_2:

$$d_2(\alpha_1, \alpha_2) = \frac{1}{2}(|\mu_{\alpha_1} - \mu_{\alpha_2}| + |\nu_{\alpha_1} - \nu_{\alpha_2}| + |\pi_{\alpha_1} - \pi_{\alpha_2}|); \tag{6}$$

- Hamming distance proposed by Grzegorzewski, d_3:

$$d_3(\alpha_1, \alpha_2) = \max\{|\mu_{\alpha_1} - \mu_{\alpha_2}|, |\nu_{\alpha_1} - \nu_{\alpha_2}|\}. \tag{7}$$

3. Dynamic intuitionistic fuzzy weighted geometric operator

Information aggregation is an essential process of gathering relevant information from multiple sources and thus is an important research topic in the field of information fusion. Atanassov [2–4] defined some basic operations and relations over IFSs. De et al. [37] developed some new operations such concentration, dilation and normalization of IFSs. Xu and Yager [20] developed some geometric operators to aggregate intuitionistic fuzzy information. All these operations, relations and operators can only be used to deal with time independent arguments. However, if time is taken into account, for example, the argument information may be collected at different periods, then the aggregation operators and their associated weights should not be kept constant. As a result, based on (4), Xu and Yager [23] defined the notion of intuitionistic fuzzy variables.

Definition 5. Let t be a time variable, then we call $\alpha(t) = (\mu_{\alpha(t)}, \nu_{\alpha(t)}, \pi_{\alpha(t)})$ an intuitionistic fuzzy variable, where

$$\mu_{\alpha(t)} \in [0, 1],\ \nu_{\alpha(t)} \in [0, 1], \quad \mu_{\alpha(t)} + \nu_{\alpha(t)} \le 1, \qquad \pi_{\alpha(t)} = 1 - \mu_{\alpha(t)} - \nu_{\alpha(t)}. \tag{8}$$

For an intuitionistic fuzzy variable $\alpha(t) = (\mu_{\alpha(t)}, \nu_{\alpha(t)}, \pi_{\alpha(t)})$, if $t = t_1, t_2, \ldots, t_p$, then $\alpha(t_1), \alpha(t_2), \ldots, \alpha(t_p)$ indicate p IFNs collected at p different periods. Below we introduce some operations related to IFNs.

Definition 6. Let $\alpha(t_1) = (\mu_{\alpha(t_1)}, \nu_{\alpha(t_1)}, \pi_{\alpha(t_1)})$ and $\alpha_2(t_2) = (\mu_{\alpha(t_2)}, \nu_{\alpha(t_2)}, \pi_{\alpha(t_2)})$ be two IFNs, then

(1) $\alpha(t_1) \otimes \alpha(t_2) = (\mu_{\alpha(t_1)}\mu_{\alpha(t_2)}, \nu_{\alpha(t_1)} + \nu_{\alpha(t_2)} - \nu_{\alpha(t_1)}\nu_{\alpha(t_2)}, (1 - \nu_{\alpha(t_1)})(1 - \nu_{\alpha(t_2)}) - \mu_{\alpha(t_1)}\mu_{\alpha(t_2)})$.
(2) $(\alpha(t_1))^\lambda = (\mu^\lambda_{\alpha(t_1)}, 1 - (1 - \nu_{\alpha(t_1)})^\lambda, (1 - \nu_{\alpha(t_1)})^\lambda - \mu^\lambda_{\alpha(t_1)}), \lambda > 0$.

[2] The notion of intuitionistic fuzzy numbers, already used in the present form in some papers [17,18,20], has two problems: in the particular fuzzy case the well-known notion of fuzzy numbers is not found again; given the possibility of confusion — another notion of intuitionistic fuzzy numbers, which in a particular case becomes a fuzzy number, was studied and used in [32,33].

From Definition 6, the operation results are also IFNs and we can get the following results:

(1) $\alpha(t_1) \otimes \alpha(t_2) = \alpha(t_2) \otimes \alpha(t_1)$.
(2) $(\alpha(t_1) \otimes \alpha(t_2))^{\lambda} = \alpha(t_1)^{\lambda} \otimes \alpha(t_2)^{\lambda},\ \lambda > 0$.
(3) $\alpha(t_1)^{\lambda_1} \otimes \alpha(t_1)^{\lambda_2} = \alpha(t_1)^{\lambda_1+\lambda_2},\ \lambda_1, \lambda_2 > 0$.

Definition 7. Let $\alpha(t_1), \alpha(t_2), \ldots, \alpha(t_p)$ be a collection of IFNs collected at p different periods t_k $(k = 1, 2, \ldots, p)$, and $\lambda(t) = (\lambda(t_1), \lambda(t_2), \ldots, \lambda(t_p))^T$ be the weight vector of periods t_k $(k = 1, 2, \ldots, p)$, then we call

$$\text{DIFWG}_{\lambda(t)}(\alpha(t_1), \alpha(t_2), \ldots, \alpha(T_p)) = \alpha(t_1)^{\lambda(t_1)} \otimes \alpha(t_2)^{\lambda(t_2)} \otimes \cdots \otimes \alpha(t_p)^{\lambda(t_p)} \tag{9}$$

a dynamic intuitionistic fuzzy weighted geometric (DIFWG) operator.

By Definition 6, (9) can be rewritten as follows:

$$\begin{aligned} &\text{DIFWG}_{\lambda(t)}(\alpha(t_1), \alpha(t_2), \ldots, \alpha(t_p)) \\ &= \left(\prod_{k=1}^{p} \mu_{\alpha(t_k)}^{\lambda(t_k)}, 1 - \prod_{k=1}^{p} (1 - \nu_{\alpha(t_k)})^{\lambda(t_k)}, \prod_{k=1}^{p} (1 - \nu_{\alpha(t_k)})^{\lambda(t_k)} - \prod_{k=1}^{p} \mu_{\alpha(t_k)}^{\lambda(t_k)} \right), \end{aligned} \tag{10}$$

where $\lambda(t_k) \geq 0$, $k = 1, 2, \ldots, p$, and $\sum_{k=1}^{p} \lambda(t_k) = 1$.

Based on Definition 7, we have the following properties.

Theorem 1. *Let $\alpha(t_1), \alpha(t_2), \ldots, \alpha(t_p)$ be a collection of IFNs collected at p different periods $t_k(k = 1, 2, \ldots, p)$ and $\lambda(t) = (\lambda(t_1), \lambda(t_2), \ldots, \lambda(t_p))^T$ is the weight vector of the periods $t_k(k = 1, 2, \ldots, p)$ with $\lambda(t_k) \geq 0$ and $\sum_{k=1}^{p} \lambda(t_k) = 1$; then we have the following.*

(1) *(Idempotency): If all $\alpha(t_k) = (\mu_{\alpha(t_k)}, \nu_{\alpha(t_k)}, \pi_{\alpha(t_k)})(k = 1, 2, \ldots, p)$ are equal, i.e., $\alpha(t_k) = \alpha(t)$ for all k, then*

$$\text{DIFWG}_{\lambda(t)}(\alpha(t_1), \alpha(t_2), \ldots, \alpha(t_p)) = \alpha(t).$$

(2) *(Boundedness):*

$$\alpha(t)^{-} \leq \text{DIFWG}_{\lambda(t)}(\alpha(t_1), \alpha(t_2), \ldots, \alpha(t_p)) \leq \alpha(t)^{+},$$

where $\alpha(t)^{-} = \min_k \alpha(t_k)$ and $\alpha(t)^{+} = \max_k \alpha(t_k)$.

(3) *(Monotonicity): Let $\alpha(t_k)^{*} = (\mu^{*}_{\alpha(t_k)}, \nu^{*}_{\alpha(t_k)}, \pi^{*}_{\alpha(t_k)})(k = 1, 2, \ldots, p)$ be a collection of IFNs. If $\alpha(t_k) \leq \alpha(t_k)^{*}$, for all k, then*

$$\text{DIFWG}_{\lambda(t)}(\alpha(t_1), \alpha(t_2), \ldots, \alpha(t_p)) \leq \text{DIFWG}_{\lambda(t)}(\alpha(t_1)^{*}, \alpha(t_2)^{*}, \ldots, \alpha(t_p)^{*}).$$

4. An approach to DIF-MADM

In this section, we consider DIF-MADM problems where all attribute values are expressed in intuitionistic fuzzy numbers, which are collected at different periods.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a discrete set of n feasible alternatives, and let $G = \{G_1, G_2, \ldots, G_m\}$ be a finite set of m attributes, whose weight vector is $w = (w_1, w_2, \ldots, w_m)^T$, where $w_j \geq 0, j = 1, 2, \ldots, m$, $\sum_{j=1}^{m} w_j = 1$. There are p periods $t_k(k = 1, 2, \ldots, p)$, whose weight vector is $\lambda(t) = (\lambda(t_1), \lambda(t_2), \ldots, \lambda(t_p))^T$, where $\lambda(t_k) \geq 0, k = 1, 2, \ldots, p$, $\sum_{k=1}^{p} \lambda(t_k) = 1$. Suppose that $R(t_k) = (r_{ij}(t_k))_{n\times m}$ is an intuitionistic fuzzy decision matrix of the period t_k, where $r_{ij}(t_k) = (\mu_{r_{ij}(t_k)}, \nu_{r_{ij}(t_k)}, \pi_{r_{ij}(t_k)})$ is an attribute value, denoted by an IFN, $\mu_{r_{ij}(t_k)}$ indicates the degree that the alternative x_i should satisfy the attribute G_j at period t_k, $\nu_{r_{ij}(t_k)}$ indicates the degree that the alternative x_i should not satisfy the attribute G_j at period t_k, and $\pi_{r_{ij}(t_k)}$ indicates the degree of indeterminacy of the alternative x_i to the attribute G_j, such that

$$\begin{aligned} &\mu_{r_{ij}(t_k)} \in [0, 1],\ \nu_{r_{ij}(t_k)} \in [0, 1], \quad \mu_{r_{ij}(t_k)} + \nu_{r_{ij}(t_k)} \leq 1, \\ &\pi_{r_{ij}(t_k)} = 1 - \mu_{r_{ij}(t_k)} - \nu_{r_{ij}(t_k)}, \quad i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m. \end{aligned} \tag{11}$$

Based on the above decision information, in what follows, we propose a practical procedure to rank and select the most alternative(s):

Procedure I. *Step* 1. Utilize the DIFWG operator:

$$\begin{aligned} r_{ij} &= \text{DIFWG}_{\lambda(t)}(r_{ij}(t_1), r_{ij}(t_2), \ldots, r_{ij}(t_p)) \\ &= \left(\prod_{k=1}^{p} \mu_{r_{ij}(t_k)}^{\lambda(t_k)}, 1 - \prod_{k=1}^{p} (1 - \nu_{r_{ij}(t_k)})^{\lambda(t_k)}, \prod_{k=1}^{p} (1 - \nu_{r_{ij}(t_k)})^{\lambda(t_k)} - \prod_{k=1}^{p} \mu_{r_{ij}(t_k)}^{\lambda(t_k)} \right) \end{aligned} \tag{12}$$

to aggregate all the intuitionistic fuzzy decision matrices $R(t_k) = (r_{ij}(t_k))_{n\times m}(k = 1, 2, \dots, p)$ into a complex intuitionistic fuzzy decision matrix $R = (r_{ij})_{n\times m}$, where $r_{ij} = (\mu_{ij}, \nu_{ij}, \pi_{ij})$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$.

Step 2. Define $\alpha^+ = (\alpha_1^+, \alpha_2^+, \dots, \alpha_m^+)^T$ as the intuitionistic fuzzy ideal solution (IFIS), where $\alpha_j^+ = \max_i r_{ij} (j = 1, 2, \dots, m)$ are the m largest IFNs. Denote α_j^+ by $\alpha_j^+ = (\mu_j^+, \nu_j^+, \pi_j^+)$. Furthermore, for convenience of depiction, we denote the alternatives $x_i (i = 1, 2, \dots, n)$ by $x_i = (r_{i1}, r_{i2}, \dots, r_{im})^T$, $i = 1, 2, \dots, n$.

Step 3. Utilize (5)–(7) to compute the values $S(x_i)$ and $R(x_i)$ for each alternative x_i $(i = 1, 2, \dots, n)$, which represent the average and the worst group scores of the alternatives x_i, respectively, with the relations

- Burillo and Bustince's method, d_1:

$$\begin{aligned} S_{d_1}(x_i) &= \sum_{j=1}^{m} w_j \cdot d_1(\alpha_j^+, r_{ij}) \\ &= \frac{1}{2}\sum_{j=1}^{m} w_j(|\mu_j^+ - \mu_{ij}| + |\nu_j^+ - \nu_{ij}|), \end{aligned} \tag{13}$$

$$\begin{aligned} R_{d_1}(x_i) &= \max_{1\le j\le m} w_j \cdot d_1(\alpha_j^+, r_{ij}) \\ &= \max_{1\le j\le m} \frac{w_j(|\mu_j^+ - \mu_{ij}| + |\nu_j^+ - \nu_{ij}|)}{2}. \end{aligned} \tag{14}$$

- Szmidt and Kacprzyk's method, d_2:

$$\begin{aligned} S_{d_2}(x_i) &= \sum_{j=1}^{m} w_j \cdot d_2(\alpha_j^+, r_{ij}) \\ &= \frac{1}{2}\sum_{j=1}^{m} w_j(|\mu_j^+ - \mu_{ij}| + |\nu_j^+ - \nu_{ij}| + |\pi_j^+ - \pi_{ij}|), \end{aligned} \tag{15}$$

$$\begin{aligned} R_{d_2}(x_i) &= \max_{1\le j\le m} w_j \cdot d_2(\alpha_j^+, r_{ij}) \\ &= \max_{1\le j\le m} \frac{w_j(|\mu_j^+ - \mu_{ij}| + |\nu_j^+ - \nu_{ij}| + |\pi_j^+ - \pi_{ij}|)}{2}. \end{aligned} \tag{16}$$

- Grzegorzewski's method, d_3:

$$\begin{aligned} S_{d_3}(x_i) &= \sum_{j=1}^{m} w_j \cdot d_h(\alpha_j^+, r_{ij}) \\ &= \sum_{j=1}^{m} w_j \cdot \max\{|\mu_j^+ - \mu_{ij}|, |\nu_j^+ - \nu_{ij}|\}, \end{aligned} \tag{17}$$

$$\begin{aligned} R_{d_3}(x_i) &= \max_{1\le j\le m} w_j \cdot d_h(\alpha_j^+, r_{ij}) \\ &= \max_{1\le j\le m} w_j \cdot \max\{|\mu_j^+ - \mu_{ij}|, |\nu_j^+ - \nu_{ij}|\}. \end{aligned} \tag{18}$$

Step 4. Compute the $Q_h(x_i)$ $(h = d_1, d_2, d_3)$ values for each alternative $x_i (i = 1, 2, \dots, n)$ with the relation

$$Q_h(x_i) = \frac{\upsilon(S_h(x_i) - S_h^*)}{S_h^- - S_h^*} + \frac{(1-\upsilon)(R_h(x_i) - R_h^*)}{R_h^- - R_h^*}, \tag{19}$$

where

$$S_h^* = \min_{1\le i\le n} S_h(x_i), \qquad S_h^- = \max_{1\le i\le n} S_h(x_i), \quad h = d_1, d_2, d_3, \tag{20}$$

$$R_h^* = \min_{1\le i\le n} R_h(x_i), \qquad R_h^- = \max_{1\le i\le n} R_h(x_i), \quad h = d_1, d_2, d_3, \tag{21}$$

and $\upsilon \in [0, 1]$ is the weight of the decision making strategy of "the majority of attribute" (or "the maximum group utility").

Step 5. Rank the alternatives by sorting each S_h, R_h and Q_h $(h = d_1, d_2, d_3)$ values in an decreasing order. The result is a set of three ranking lists.

Step 6. Propose the alternative $x' \in X$ which is ranked the best by $Q_h(x_i)$ $(h = d_1, d_2, d_3)$ (where $Q_h(x') = \min_{1\le i\le n}\{Q_h(x_i)\}$) as compromise solution if the following two conditions (a) and (b) are satisfied

(a) "Acceptable advantage":

$$Q_h(x'') - Q_h(x') \geq \frac{1}{(m-1)}$$

where $x'' \in X$ is the alternative in the second position in the list ranked by $Q_h(x_i)$.

(b) "Acceptable stability in decision making": x' must also be the best ranked by $S_h(x_i)$ or/and $R_h(x_i)$. This compromise solution is stable within a decision making process, which could be: "voting by majority rule" (when $\upsilon > 0.5$), or "by consensus" (when $\upsilon = 0.5$), or "with veto" (when $\upsilon < 0.5$).

If one of the above conditions is not satisfied, then a set of compromise solutions is proposed, which consists of:

- Alternatives x' and x'' if only the condition (b) is not satisfied, or
- Alternatives $x', x'', \ldots, x^N$ if the condition (a) is not satisfied, where N is the largest i such that $Q_h(x^i) - Q_h(x') < \frac{1}{(m-1)}$, which means that the positions of these alternatives are in closeness.

Step 7. End.

5. An approach to DIF-MADM under interval uncertainty

Atanassov and Gargov [38] generalized IFS and defined the notion of the interval-valued IFS (IVIFS), which is characterized by a membership function and a non-membership function whose values are intervals rather than exact numbers.

Definition 8. Let X be a fixed set, an IVIFS $\tilde{A}$ in X is an object having the form:

$$\tilde{A} = \{\langle x, \tilde{\mu}_{\tilde{A}}(x), \tilde{\nu}_{\tilde{A}}(x)\rangle \mid x \in X\}, \tag{22}$$

where $\tilde{\mu}_{\tilde{A}}(x) = [\tilde{\mu}^L_{\tilde{A}}(x), \tilde{\mu}^U_{\tilde{A}}(x)] \subset [0, 1]$ and $\tilde{\nu}_{\tilde{A}}(x) = [\tilde{\nu}^L_{\tilde{A}}(x), \tilde{\nu}^U_{\tilde{A}}(x)] \subset [0, 1]$ are intervals, $\tilde{\mu}^L_{\tilde{A}}(x) = \inf \tilde{\mu}_{\tilde{A}}(x)$, $\tilde{\mu}^U_{\tilde{A}}(x) = \sup \tilde{\mu}_{\tilde{A}}(x)$, $\tilde{\nu}^L_{\tilde{A}}(x) = \inf \tilde{\nu}_{\tilde{A}}(x)$, $\tilde{\nu}^U_{\tilde{A}}(x) = \sup \tilde{\nu}_{\tilde{A}}(x)$, and for every $x \in X$:

$$\tilde{\mu}^U_{\tilde{A}}(x) + \tilde{\nu}^U_{\tilde{A}}(x) \leq 1. \tag{23}$$

Let $\tilde{\pi}_{\tilde{A}}(x) = [\tilde{\pi}^L_{\tilde{A}}(x), \tilde{\pi}^U_{\tilde{A}}(x)]$, where

$$\tilde{\pi}^L_{\tilde{A}}(x) = 1 - \tilde{\mu}^U_{\tilde{A}}(x) - \tilde{\nu}^U_{\tilde{A}}(x), \qquad \tilde{\pi}^U_{\tilde{A}}(x) = 1 - \tilde{\mu}^U_{\tilde{A}}(x) - \tilde{\nu}^U_{\tilde{A}}(x), \quad \text{for all } x \in X. \tag{24}$$

Xu and Yager [23] called the triple $(\tilde{\mu}_{\tilde{A}}(x), \tilde{\nu}_{\tilde{A}}(x), \tilde{\pi}_{\tilde{A}}(x))$ an interval-valued intuitionistic fuzzy number (IVIFN). For convenience, we denote an IVIFN by $\tilde{\alpha} = (\tilde{\mu}_{\tilde{\alpha}}, \tilde{\nu}_{\tilde{\alpha}}, \tilde{\pi}_{\tilde{\alpha}})$, where

$$\begin{aligned} &\tilde{\mu}_{\tilde{\alpha}} = [\tilde{\mu}^L_{\tilde{\alpha}}, \tilde{\mu}^U_{\tilde{\alpha}}] \subset [0, 1], \qquad \tilde{\nu}_{\tilde{\alpha}} = [\tilde{\nu}^L_{\tilde{\alpha}}, \tilde{\nu}^U_{\tilde{\alpha}}] \subset [0, 1], \qquad \tilde{\mu}^U_{\tilde{\alpha}} + \tilde{\nu}^U_{\tilde{\alpha}} \leq 1, \\ &\tilde{\pi}_{\tilde{\alpha}} = [\tilde{\pi}^L_{\tilde{\alpha}}, \tilde{\pi}^U_{\tilde{\alpha}}] = [1 - \tilde{\mu}^U_{\tilde{\alpha}} - \tilde{\nu}^U_{\tilde{\alpha}}, 1 - \tilde{\mu}^L_{\tilde{\alpha}} - \tilde{\nu}^L_{\tilde{\alpha}}]. \end{aligned} \tag{25}$$

Obviously, by (25), we know that ([1, 1], [0, 0], [0, 0]) and ([0, 0], [1, 1], [0, 0]) are the largest and smallest IVIFNs, respectively.

Based on the score function [39] and the accuracy function [40] of IVIFNs, we define a method to compare two IVIFNs as follows:

Definition 9. Let $\tilde{\alpha}_1 = ([\tilde{\mu}^L_{\tilde{\alpha}_1}, \tilde{\mu}^U_{\tilde{\alpha}_1}], [\tilde{\nu}^L_{\tilde{\alpha}_1}, \tilde{\nu}^U_{\tilde{\alpha}_1}], [\tilde{\pi}^L_{\tilde{\alpha}_1}, \tilde{\pi}^U_{\tilde{\alpha}_1}])$ and $\tilde{\alpha}_2 = ([\tilde{\mu}^L_{\tilde{\alpha}_2}, \tilde{\mu}^U_{\tilde{\alpha}_2}], [\tilde{\nu}^L_{\tilde{\alpha}_2}, \tilde{\nu}^U_{\tilde{\alpha}_2}], [\tilde{\pi}^L_{\tilde{\alpha}_2}, \tilde{\pi}^U_{\tilde{\alpha}_2}])$ be two IVIFNs, $s(\tilde{\alpha}_1) = \frac{1}{2}(\tilde{\mu}^L_{\tilde{\alpha}_1} - \tilde{\nu}^L_{\tilde{\alpha}_1} + \tilde{\mu}^U_{\tilde{\alpha}_1} - \tilde{\nu}^U_{\tilde{\alpha}_1})$ and $s(\tilde{\alpha}_2) = \frac{1}{2}(\tilde{\mu}^L_{\tilde{\alpha}_2} - \tilde{\nu}^L_{\tilde{\alpha}_2} + \tilde{\mu}^U_{\tilde{\alpha}_2} - \tilde{\nu}^U_{\tilde{\alpha}_2})$ be the score of $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$, respectively, and $h(\tilde{\alpha}_1) = \frac{1}{2}(\tilde{\mu}^L_{\tilde{\alpha}_1} + \tilde{\nu}^L_{\tilde{\alpha}_1} + \tilde{\mu}^U_{\tilde{\alpha}_1} + \tilde{\nu}^U_{\tilde{\alpha}_1})$ and $h(\tilde{\alpha}_2) = \frac{1}{2}(\tilde{\mu}^L_{\tilde{\alpha}_2} + \tilde{\nu}^L_{\tilde{\alpha}_2} + \tilde{\mu}^U_{\tilde{\alpha}_2} + \tilde{\nu}^U_{\tilde{\alpha}_2})$ be the accuracy degree of $\tilde{a}_1$ and $\tilde{a}_2$, respectively, then:

- if $s(\tilde{\alpha}_1) < s(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1$ is smaller than $\tilde{\alpha}_2$, denoted by $\tilde{\alpha}_1 < \tilde{\alpha}_2$;
- if $s(\tilde{\alpha}_1) = s(\tilde{\alpha}_2)$, then
  (1) if $h(\tilde{\alpha}_1) = h(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ represent the same information, i.e., $\tilde{\mu}_{\tilde{\alpha}_1} = \tilde{\mu}_{\tilde{\alpha}_2}$, $\tilde{\nu}_{\tilde{\alpha}_1} = \tilde{\nu}_{\tilde{\alpha}_2}$, and $\tilde{\pi}_{\tilde{\alpha}_1} = \tilde{\pi}_{\tilde{\alpha}_2}$, denoted by $\tilde{\alpha}_1 = \tilde{\alpha}_2$;
  (2) if $h(\tilde{\alpha}_1) < h(\tilde{\alpha}_2)$, then $\tilde{\alpha}_1$ is smaller than $\tilde{\alpha}_2$, denoted by $\tilde{\alpha}_1 < \tilde{\alpha}_2$.

In what follows, we extend Definition 4 to define three types of the Hamming distance between two IVIFNs as follows:

Definition 10. Let $\tilde{\alpha}_1 = ([\tilde{\mu}^L_{\tilde{\alpha}_1}, \tilde{\mu}^U_{\tilde{\alpha}_1}], [\tilde{\nu}^L_{\tilde{\alpha}_1}, \tilde{\nu}^U_{\tilde{\alpha}_1}], [\tilde{\pi}^L_{\tilde{\alpha}_1}, \tilde{\pi}^U_{\tilde{\alpha}_1}])$ and $\tilde{\alpha}_2 = ([\tilde{\mu}^L_{\tilde{\alpha}_2}, \tilde{\mu}^U_{\tilde{\alpha}_2}], [\tilde{\nu}^L_{\tilde{\alpha}_2}, \tilde{\nu}^U_{\tilde{\alpha}_2}], [\tilde{\pi}^L_{\tilde{\alpha}_2}, \tilde{\pi}^U_{\tilde{\alpha}_2}])$ be IVIFNs, then

- Hamming distance by an extension of Burillo and Bustince's method, d_1:

$$d_1(\tilde{\alpha}_1, \tilde{\alpha}_2) = \frac{1}{4}\left(|\tilde{\mu}^L_{\tilde{\alpha}_1} - \tilde{\mu}^L_{\tilde{\alpha}_2}| + |\tilde{\mu}^U_{\tilde{\alpha}_1} - \tilde{\mu}^U_{\tilde{\alpha}_2}| + |\tilde{\nu}^L_{\tilde{\alpha}_1} - \tilde{\nu}^L_{\tilde{\alpha}_2}| + |\tilde{\nu}^U_{\tilde{\alpha}_1} - \tilde{\nu}^U_{\tilde{\alpha}_2}|\right); \tag{26}$$

- Hamming distance by an extension of Szmidt and Kacprzyk's method, d_2:

$$d_2(\tilde{\alpha}_1, \tilde{\alpha}_2) = \frac{1}{4}\left(|\tilde{\mu}^L_{\tilde{\alpha}_1} - \tilde{\mu}^L_{\tilde{\alpha}_2}| + |\tilde{\mu}^U_{\tilde{\alpha}_1} - \tilde{\mu}^U_{\tilde{\alpha}_2}| + |\tilde{\nu}^L_{\tilde{\alpha}_1} - \tilde{\nu}^L_{\tilde{\alpha}_2}| + |\tilde{\nu}^U_{\tilde{\alpha}_1} - \tilde{\nu}^U_{\tilde{\alpha}_2}| + |\tilde{\pi}^L_{\tilde{\alpha}_1} - \tilde{\pi}^L_{\tilde{\alpha}_2}| + |\tilde{\pi}^U_{\tilde{\alpha}_1} - \tilde{\pi}^U_{\tilde{\alpha}_2}|\right); \tag{27}$$

- Hamming distance by an extension of Grzegorzewski's method, d_3:

$$d_3(\tilde{\alpha}_1, \tilde{\alpha}_2) = \frac{1}{2}\left(\max\{|\tilde{\mu}^L_{\tilde{\alpha}_1} - \tilde{\mu}^L_{\tilde{\alpha}_2}|, |\tilde{\mu}^U_{\tilde{\alpha}_1} - \tilde{\mu}^U_{\tilde{\alpha}_2}|\} + \max\{|\tilde{\nu}^L_{\tilde{\alpha}_1} - \tilde{\nu}^L_{\tilde{\alpha}_2}|, |\tilde{\nu}^U_{\tilde{\alpha}_1} - \tilde{\nu}^U_{\tilde{\alpha}_2}|\}\right). \tag{28}$$

From Definitions 5–7, we extend those to the case of IVIFNs.

Definition 11. Let t be a time variable, then we call $\tilde{\alpha}(t) = \left([\tilde{\mu}^L_{\tilde{\alpha}(t)}, \tilde{\mu}^U_{\tilde{\alpha}(t)}], [\tilde{\nu}^L_{\tilde{\alpha}(t)}, \tilde{\nu}^U_{\tilde{\alpha}(t)}], [\tilde{\pi}^L_{\tilde{\alpha}(t)}, \tilde{\pi}^U_{\tilde{\alpha}(t)}]\right)$ an uncertain intuitionistic fuzzy variable, where

$$\begin{aligned} &\tilde{\mu}_{\tilde{\alpha}(t)} = [\tilde{\mu}^L_{\tilde{\alpha}(t)}, \tilde{\mu}^U_{\tilde{\alpha}(t)}] \subset [0,1], \qquad \tilde{\nu}_{\tilde{\alpha}(t)} = [\tilde{\nu}^L_{\tilde{\alpha}(t)}, \tilde{\nu}^U_{\tilde{\alpha}(t)}] \subset [0,1], \qquad \tilde{\mu}^U_{\tilde{\alpha}(t)} + \tilde{\nu}^U_{\tilde{\alpha}(t)} \le 1, \\ &\tilde{\pi}_{\tilde{\alpha}(t)} = [\tilde{\pi}^L_{\tilde{\alpha}(t)}, \tilde{\pi}^U_{\tilde{\alpha}(t)}] = [1 - \tilde{\mu}^U_{\tilde{\alpha}(t)} - \tilde{\nu}^U_{\tilde{\alpha}(t)}, 1 - \tilde{\mu}^L_{\tilde{\alpha}(t)} - \tilde{\nu}^L_{\tilde{\alpha}(t)}]. \end{aligned} \tag{29}$$

For an uncertain intuitionistic fuzzy variable $\tilde{\alpha}(t) = \left([\tilde{\mu}^L_{\tilde{\alpha}(t)}, \tilde{\mu}^U_{\tilde{\alpha}(t)}], [\tilde{\nu}^L_{\tilde{\alpha}(t)}, \tilde{\nu}^U_{\tilde{\alpha}(t)}], [\tilde{\pi}^L_{\tilde{\alpha}(t)}, \tilde{\pi}^U_{\tilde{\alpha}(t)}]\right)$, if $t = t_1, t_2, \dots, t_p$, then $\tilde{\alpha}(t_1), \tilde{\alpha}(t_2), \dots, \tilde{\alpha}(t_p)$ indicate p IVIFNs collected at p different periods.

Now we introduce the following operations related to IVIFNs.

Definition 12. Let $\tilde{\alpha}(t_k) = \left([\tilde{\mu}^L_{\tilde{\alpha}(t_k)}, \tilde{\mu}^U_{\tilde{\alpha}(t+k)}], [\tilde{\nu}^L_{\tilde{\alpha}(t_k)}, \tilde{\nu}^U_{\tilde{\alpha}(t_k)}], [\tilde{\pi}^L_{\tilde{\alpha}(t_k)}, \tilde{\pi}^U_{\tilde{\alpha}(t_k)}]\right)(k = 1, 2)$ be two IVIFNs, then

(1) $\tilde{\alpha}(t_1) \otimes \tilde{\alpha}(t_2) = \left(\left[\tilde{\mu}^L_{\tilde{\alpha}(t_1)}\tilde{\mu}^L_{\tilde{\alpha}(t_2)}, \tilde{\mu}^U_{\tilde{\alpha}(t_1)}\tilde{\mu}^U_{\tilde{\alpha}(t_2)}\right], \left[\tilde{\nu}^L_{\tilde{\alpha}(t_1)} + \tilde{\nu}^L_{\tilde{\alpha}(t_2)} - \tilde{\nu}^L_{\tilde{\alpha}(t_1)}\tilde{\nu}^L_{\tilde{\alpha}(t_2)}, \tilde{\nu}^U_{\tilde{\alpha}(t_1)} + \tilde{\nu}^U_{\tilde{\alpha}(t_2)} - \tilde{\nu}^U_{\tilde{\alpha}(t_1)}\tilde{\nu}^U_{\tilde{\alpha}(t_2)}\right], \left[(1 - \tilde{\nu}^L_{\tilde{\alpha}(t_1)})(1 - \tilde{\nu}^L_{\tilde{\alpha}(t_2)}) - \tilde{\mu}^L_{\tilde{\alpha}(t_1)}\tilde{\mu}^L_{\tilde{\alpha}(t_2)}, (1 - \tilde{\nu}^U_{\tilde{\alpha}(t_1)})(1 - \tilde{\nu}^U_{\tilde{\alpha}(t_2)}) - \tilde{\mu}^U_{\tilde{\alpha}(t_1)}\tilde{\mu}^U_{\tilde{\alpha}(t_2)}\right]\right)$.

(2) $\tilde{\alpha}(t_1)^\lambda = \left(\left[(\tilde{\mu}^L_{\tilde{\alpha}(t_1)})^\lambda, (\tilde{\mu}^U_{\tilde{\alpha}(t_1)})^\lambda\right], \left[1 - (1 - \tilde{\nu}^L_{\tilde{\alpha}(t_1)})^\lambda, 1 - (1 - \tilde{\nu}^U_{\tilde{\alpha}(t_1)})^\lambda\right], \left[(1 - \tilde{\nu}^L_{\tilde{\alpha}(t_1)})^\lambda - (\tilde{\mu}^L_{\tilde{\alpha}(t_1)})^\lambda, (1 - \tilde{\nu}^U_{\tilde{\alpha}(t_1)})^\lambda - (\tilde{\mu}^U_{\tilde{\alpha}(t_1)})^\lambda\right]\right),\ \lambda > 0$.

Definition 13. Let $\tilde{\alpha}(t_1), \tilde{\alpha}(t_2), \dots, \tilde{\alpha}(t_p)$ be a collection of IVIFNs collected at p different periods $t_k (k = 1, 2, \dots, p)$, and $\lambda(t) = (\lambda(t_1), \lambda(t_2), \dots, \lambda(t_p))^T$ be the weight vector of periods t_k $(k = 1, 2, \dots, p)$, then we call

$$\mathrm{UDIFWG}_{\lambda(t)}(\tilde{\alpha}(t_1), \tilde{\alpha}(t_2), \dots, \tilde{\alpha}(t_p)) = \tilde{\alpha}(t_1)^{\lambda(t_1)} \otimes \tilde{\alpha}(t_2)^{\lambda(t_2)} \otimes \cdots \otimes \tilde{\alpha}(t_p)^{\lambda(t_p)} \tag{30}$$

an uncertain dynamic intuitionistic fuzzy weighted geometric (UDIFWG) operator, which can be rewritten as follows:

$$\begin{aligned} \mathrm{UDIFWG}_{\lambda(t)}(\tilde{\alpha}(t_1), \tilde{\alpha}(t_2), \dots, \tilde{\alpha}(t_p)) = &\left(\left[\prod_{k=1}^p (\tilde{\mu}^L_{\tilde{\alpha}(t_k)})^{\lambda(t_k)}, \prod_{k=1}^p (\tilde{\mu}^U_{\tilde{\alpha}(t_k)})^{\lambda(t_k)}\right],\right. \\ &\left[1 - \prod_{k=1}^p (1 - \tilde{\nu}^L_{\tilde{\alpha}(t_k)})^{\lambda(t_k)}, 1 - \prod_{k=1}^p (1 - \tilde{\nu}^U_{\tilde{\alpha}(t_k)})^{\lambda(t_k)}\right], \\ &\left.\left[\prod_{k=1}^p (1 - \tilde{\nu}^L_{\tilde{\alpha}(t_k)})^{\lambda(t_k)} - \prod_{k=1}^p (\tilde{\mu}^L_{\tilde{\alpha}(t_k)})^{\lambda(t_k)}, \prod_{k=1}^p (1 - \tilde{\nu}^U_{\tilde{\alpha}(t_k)})^{\lambda(t_k)} - \prod_{k=1}^p (\tilde{\mu}^U_{\tilde{\alpha}(t_k)})^{\lambda(t_k)}\right]\right), \end{aligned} \tag{31}$$

where $\lambda(t_k) \ge 0$, $k = 1, 2, \dots, p$, and $\sum_{k=1}^p \lambda(t_k) = 1$.

Now we consider the DIF-MADM problems under interval uncertainty where all the attribute values are expressed in IVIFNs, which are collected at different periods. The following notations are used to depict the considered problems:

Let X, G, w and $\lambda(t)$ be presented as in Section 4, and let $\tilde{R}(t_k) = (\tilde{r}_{ij}(t_k))_{n \times m}$ be an uncertain intuitionistic fuzzy decision matrix of the period t_k, where $\tilde{r}_{ij}(t_k) = ([\tilde{\mu}^L_{\tilde{r}_{ij}(t_k)}, \tilde{\mu}^U_{\tilde{r}_{ij}(t_k)}], [\tilde{\nu}^L_{\tilde{r}_{ij}(t_k)}, \tilde{\nu}^U_{\tilde{r}_{ij}(t_k)}], [\tilde{\pi}^L_{\tilde{r}_{ij}(t_k)}, \tilde{\pi}^U_{\tilde{r}_{ij}(t_k)}])$ is an attribute value, denoted by an IVIFN, $[\tilde{\mu}^L_{\tilde{r}_{ij}(t_k)}, \tilde{\mu}^U_{\tilde{r}_{ij}(t_k)}]$ indicates the uncertain degree that the alternative x_i should satisfy the attribute G_j at period

t_k, $[\tilde{\nu}^L_{\tilde{r}_{ij}(t_k)}, \tilde{\nu}^U_{\tilde{r}_{ij}(t_k)}]$ indicates the uncertain degree that the alternative x_i should not satisfy the attribute G_j at period t_k, and $[\tilde{\pi}^L_{\tilde{r}_{ij}(t_k)}, \tilde{\pi}^U_{\tilde{r}_{ij}(t_k)}]$ indicates the range of indeterminacy of the alternative x_i to the attribute G_j, such that

$$\begin{aligned}&[\tilde{\mu}^L_{\tilde{r}_{ij}(t_k)}, \tilde{\mu}^U_{\tilde{r}_{ij}(t_k)}] \subset [0,1], \qquad [\tilde{\nu}^L_{\tilde{r}_{ij}(t_k)}, \tilde{\nu}^U_{\tilde{r}_{ij}(t_k)}] \subset [0,1], \qquad \tilde{\mu}^U_{\tilde{r}_{ij}(t_k)} + \tilde{\nu}^U_{\tilde{r}_{ij}(t_k)} \le 1,\\ &[\tilde{\pi}^L_{\tilde{r}_{ij}(t_k)}, \tilde{\pi}^U_{\tilde{r}_{ij}(t_k)}] = [1-\tilde{\mu}^U_{\tilde{r}_{ij}(t_k)} - \tilde{\nu}^U_{\tilde{r}_{ij}(t_k)}, 1-\tilde{\mu}^L_{\tilde{r}_{ij}(t_k)} - \tilde{\nu}^L_{\tilde{r}_{ij}(t_k)}], \quad i=1,2,\ldots,n,\ j=1,2,\ldots,m.\end{aligned} \tag{32}$$

Similar to Section 4, a procedure for solving the above problems can be described as follows:

Procedure II. *Step* 1. Utilize the UDIFWG operator:

$$\begin{aligned}\tilde{r}_{ij} &= \mathrm{UDIFWG}_{\lambda(t)}(\tilde{r}_{ij}(t_1), \tilde{r}_{ij}(t_2), \ldots, \tilde{r}_{ij}(t_p))\\ &= \left(\left[\prod_{k=1}^{p}(\tilde{\mu}^L_{\tilde{r}_{ij}(t_k)})^{\lambda(t_k)}, \prod_{k=1}^{p}(\tilde{\mu}^U_{\tilde{r}_{ij}(t_k)})^{\lambda(t_k)}\right], \left[1-\prod_{k=1}^{p}(1-\tilde{\nu}^L_{\tilde{r}_{ij}(t_k)})^{\lambda(t_k)},\right.\right.\\ &\quad \left. 1-\prod_{k=1}^{p}(1-\tilde{\nu}^U_{\tilde{r}_{ij}(t_k)})^{\lambda(t_k)}\right], \left[\prod_{k=1}^{p}(1-\tilde{\nu}^U_{\tilde{r}_{ij}(t_k)})^{\lambda(t_k)} - \prod_{k=1}^{p}(\tilde{\mu}^U_{\tilde{r}_{ij}(t_k)})^{\lambda(t_k)},\right.\\ &\quad \left.\left.\prod_{k=1}^{p}(1-\tilde{\nu}^L_{\tilde{r}_{ij}(t_k)})^{\lambda(t_k)} - \prod_{k=1}^{p}(\tilde{\mu}^L_{\tilde{r}_{ij}(t_k)})^{\lambda(t_k)}\right]\right)\end{aligned} \tag{33}$$

to aggregate all the uncertain intuitionistic fuzzy decision matrices $\tilde{R}(t_k) = (\tilde{r}_{ij}(t_k))_{n\times m}(k=1,2,\ldots,p)$ into a complex uncertain intuitionistic fuzzy decision matrix $\tilde{R} = (\tilde{r}_{ij})_{n\times m}$, where $\tilde{r}_{ij} = ([\tilde{\mu}^L_{ij}, \tilde{\mu}^L_{ij}], [\tilde{\nu}^L_{ij}, \tilde{\nu}^L_{ij}], [\tilde{\pi}^L_{ij}, \tilde{\pi}^U_{ij}])$, $i=1,2,\ldots,n$, $j=1,2,\ldots,m$.

Step 2. Define $\tilde{\alpha}^+ = (\tilde{\alpha}^+_1, \tilde{\alpha}^+_2, \ldots, \tilde{\alpha}^+_m)^T$ as the uncertain intuitionistic fuzzy ideal solution (UIFIS), where $\tilde{\alpha}^+_j = \max_i \tilde{r}_{ij} (j=1,2,\ldots,m)$ are the m largest IVIFNs. Denote $\tilde{\alpha}^+_j$ by $\tilde{\alpha}^+_j = ([\tilde{\mu}^{+L}_j, \tilde{\mu}^{+U}_j], [\tilde{\nu}^{+L}_j, \tilde{\nu}^{+U}_j], [\tilde{\pi}^{+L}_j, \tilde{\pi}^{+L}_j])$. Furthermore, for convenience of depiction, we denote the alternatives $x_i (i=1,2,\ldots,n)$ by $x_i = (\tilde{r}_{i1}, \tilde{r}_{i2}, \ldots, \tilde{r}_{im})^T$, $i=1,2,\ldots,n$.

Step 3. Utilize (26)–(28) to compute the values $\tilde{S}(x_i)$ and $\tilde{R}(x_i)$ for each alternative x_i $(i=1,2,\ldots,n)$, which represent the average and the worst group scores of the alternatives x_i, respectively, with the relations

- The extension of Burillo and Bustince's method, d_1:

$$\begin{aligned}\tilde{S}_{d_1}(x_i) &= \sum_{j=1}^{m} w_j \cdot d_1(\tilde{\alpha}^+_j, \tilde{r}_{ij})\\ &= \frac{1}{4}\sum_{j=1}^{m} w_j \left(|\tilde{\mu}^{+L}_j - \tilde{\mu}^L_{ij}| + |\tilde{\mu}^{+U}_j - \tilde{\mu}^U_{ij}| + |\tilde{\nu}^{+L}_j - \tilde{\nu}^L_{ij}| + |\tilde{\nu}^{+U}_j - \tilde{\nu}^U_{ij}|\right),\end{aligned} \tag{34}$$

$$\begin{aligned}\tilde{R}_{d_1}(x_i) &= \max_{1\le j\le m} w_j \cdot d_1(\tilde{\alpha}^+_j, \tilde{r}_{ij})\\ &= \frac{1}{4}\max_{1\le j\le m} w_j \left(|\tilde{\mu}^{+L}_j - \tilde{\mu}^L_{ij}| + |\tilde{\mu}^{+U}_j - \tilde{\mu}^U_{ij}| + |\tilde{\nu}^{+L}_j - \tilde{\nu}^L_{ij}| + |\tilde{\nu}^{+U}_j - \tilde{\nu}^U_{ij}|\right).\end{aligned} \tag{35}$$

- The extension of Szmidt and Kacprzyk's method, d_2:

$$\begin{aligned}\tilde{S}_{d_2}(x_i) &= \sum_{j=1}^{m} w_j \cdot d_2(\tilde{\alpha}^+_j, \tilde{r}_{ij})\\ &= \frac{1}{4}\sum_{j=1}^{m} w_j \big(|\tilde{\mu}^{+L}_j - \tilde{\mu}^L_{ij}| + |\tilde{\mu}^{+U}_j - \tilde{\mu}^U_{ij}| + |\tilde{\nu}^{+L}_j - \tilde{\nu}^L_{ij}| + |\tilde{\nu}^{+U}_j - \tilde{\nu}^U_{ij}|\\ &\quad + |\tilde{\pi}^{+L}_j - \tilde{\pi}^L_{ij}| + |\tilde{\pi}^{+U}_j - \tilde{\pi}^U_{ij}|\big),\end{aligned} \tag{36}$$

$$\begin{aligned}\tilde{R}_{d_2}(x_i) &= \max_{1\le j\le m} w_j \cdot d_2(\tilde{\alpha}^+_j, \tilde{r}_{ij})\\ &= \frac{1}{4}\max_{1\le j\le m} w_j \big(|\tilde{\mu}^{+L}_j - \tilde{\mu}^L_{ij}| + |\tilde{\mu}^{+U}_j - \tilde{\mu}^U_{ij}| + |\tilde{\nu}^{+L}_j - \tilde{\nu}^L_{ij}| + |\tilde{\nu}^{+U}_j - \tilde{\nu}^U_{ij}|\\ &\quad + |\tilde{\pi}^{+L}_j - \tilde{\pi}^L_{ij}| + |\tilde{\pi}^{+U}_j - \tilde{\pi}^U_{ij}|\big).\end{aligned} \tag{37}$$

- The extension of Grzegorzewski's method, d_3:

$$\begin{aligned}\tilde{S}_{d_3}(x_i) &= \sum_{j=1}^{m} w_j \cdot d_h(\tilde{\alpha}_j^+, \tilde{r}_{ij})\\ &= \frac{1}{2}\sum_{j=1}^{m} w_j \left(\max\{|\tilde{\mu}_j^{+L} - \tilde{\mu}_{ij}^L|, |\tilde{\mu}_j^{+U} - \tilde{\mu}_{ij}^U|\} + \max\{|\tilde{\nu}_j^{+L} - \tilde{\nu}_{ij}^L|, |\tilde{\nu}_j^{+U} - \tilde{\nu}_{ij}^U|\}\right), \qquad (38)\\ \tilde{R}_{d_3}(x_i) &= \max_{1\le j\le m} w_j \cdot d_h(\tilde{\alpha}_j^+, \tilde{r}_{ij})\\ &= \frac{1}{2}\max_{1\le j\le m} w_j \left(\max\{|\tilde{\mu}_j^{+L} - \tilde{\mu}_{ij}^L|, |\tilde{\mu}_j^{+U} - \tilde{\mu}_{ij}^U|\} + \max\{|\tilde{\nu}_j^{+L} - \tilde{\nu}_{ij}^L|, |\tilde{\nu}_j^{+U} - \tilde{\nu}_{ij}^U|\}\right). \qquad (39)\end{aligned}$$

Step 4. Compute the $\tilde{Q}_h(x_i)$ $(h = d_1, d_2, d_3)$ values for each alternative $x_i (i = 1, 2, \ldots, n)$ with the relation

$$\tilde{Q}_h(x_i) = \frac{v(\tilde{S}_h(x_i) - \tilde{S}_h^*)}{\tilde{S}_h^- - \tilde{S}_h^*} + \frac{(1-v)(\tilde{R}_h(x_i) - \tilde{R}_h^*)}{\tilde{R}_h^- - \tilde{R}_h^*}, \qquad (40)$$

where

$$\tilde{S}_h^* = \min_{1\le i\le n} \tilde{S}_h(x_i), \qquad \tilde{S}_h^- = \max_{1\le i\le n} \tilde{S}_h(x_i), \quad h = d_1, d_2, d_3, \qquad (41)$$

$$\tilde{R}_h^* = \min_{1\le i\le n} \tilde{R}_h(x_i), \qquad \tilde{R}_h^- = \max_{1\le i\le n} \tilde{R}_h(x_i), \quad h = d_1, d_2, d_3, \qquad (42)$$

and $v \in [0, 1]$ is the weight of decision making strategy of "the majority of attribute" (or "the maximum group utility").

Step 5. Rank the alternatives by sorting each $\tilde{S}_h$, $\tilde{R}_h$ and $\tilde{Q}_h$ $(h = d_1, d_2, d_3)$ values in decreasing order. The result is a set of three ranking lists.

Step 6. Propose the alternative $x' \in X$ which is ranked the best by $\tilde{Q}_h(x_i)$ $(h = d_1, d_2, d_3)$ (where $\tilde{Q}_h(x') = \min_{1\le i\le n}\{\tilde{Q}_h(x_i)\}$) as the compromise solution if the following two conditions (a) and (b) are satisfied

(a) "Acceptable advantage":

$$\tilde{Q}_h(x'') - \tilde{Q}_h(x') \geq \frac{1}{(m-1)}$$

where $x'' \in X$ is the alternative in the second position in the list ranked by $\tilde{Q}_h(x_i)$.

(b) "Acceptable stability in decision making": x' must also be the best ranked by $\tilde{S}_h(x_i)$ or/and $\tilde{R}_h(x_i)$. This compromise solution is stable within a decision making process, which could be: "voting by majority rule" (when $v > 0.5$), or "by consensus" (when $v = 0.5$), or "with veto" (when $v < 0.5$).

If one of the above conditions is not satisfied, then a set of compromise solutions is proposed, which consists of:

- Alternatives x' and x'' if only condition (b) is not satisfied, or
- Alternatives $x', x'', \ldots, x^N$ if condition (a) is not satisfied, where N is the largest i such that $\tilde{Q}_h(x^i) - \tilde{Q}_h(x') < \frac{1}{(m-1)}$, which means that the positions of these alternatives are in closeness.

Step 7. End.

6. Illustrative example

In this section, a problem of evaluating university faculty for tenure and promotion (adapted from Bryson and Mobolurin [41]) is used to illustrate the developed approach.

A practical use of the proposed approach involves the evaluation of university faculty for tenure and promotion. The attributes at some university are G_1: teaching, G_2: research, and G_3: service. The committee evaluates the performance of five faculty candidates (alternatives) $x_i (i = 1, 2, 3, 4, 5)$ in the three years t_k $(k = 1, 2, 3)$ according to the attributes $G_j (j = 1, 2, 3)$, and construct, respectively, the intuitionistic fuzzy decision matrices $R(t_k)$ $(k = 1, 2, 3)$ as listed in Tables 1–3. Let $\lambda(t) = (0.2, 0.3, 0.5)^T$ be the weight vector of the years t_k $(k = 1, 2, 3)$, and $w = (0.3, 0.4, 0.3)^T$ be the weight vector of the attributes G_j $(j = 1, 2, 3)$.

Now we utilize the proposed Procedure I to prioritize these faculty candidates:

Step 1. Utilize the DIFWG operator (10) to aggregate all the intuitionistic fuzzy decision matrices $R(t_k)$ into a complex intuitionistic fuzzy decision matrix R (see Table 4).

Table 1
Intuitionistic fuzzy decision matrix $R(t_1)$.

	G_1	G_2	G_3
x_1	(0.8, 0.1, 0.1)	(0.9, 0.1, 0.0)	(0.7, 0.2, 0.1)
x_2	(0.7, 0.3, 0.0)	(0.6, 0.2, 0.2)	(0.6, 0.3, 0.1)
x_3	(0.5, 0.4, 0.1)	(0.7, 0.3, 0.0)	(0.6, 0.3, 0.1)
x_4	(0.9, 0.1, 0.0)	(0.7, 0.2, 0.1)	(0.8, 0.2, 0.0)
x_5	(0.6, 0.1, 0.3)	(0.8, 0.2, 0.0)	(0.5, 0.1, 0.4)

Table 2
Intuitionistic fuzzy decision matrix $R(t_2)$.

	G_1	G_2	G_3
x_1	(0.9, 0.1, 0.0)	(0.8, 0.2, 0.0)	(0.8, 0.1, 0.1)
x_2	(0.8, 0.2, 0.0)	(0.5, 0.1, 0.4)	(0.7, 0.2, 0.1)
x_3	(0.5, 0.5, 0.0)	(0.7, 0.2, 0.1)	(0.8, 0.2, 0.1)
x_4	(0.9, 0.1, 0.0)	(0.9, 0.1, 0.0)	(0.7, 0.3, 0.0)
x_5	(0.5, 0.2, 0.3)	(0.6, 0.3, 0.1)	(0.6, 0.2, 0.2)

Table 3
Intuitionistic fuzzy decision matrix $R(t_3)$.

	G_1	G_2	G_3
x_1	(0.7, 0.1, 0.2)	(0.9, 0.1, 0.0)	(0.7, 0.1, 0.2)
x_2	(0.5, 0.1, 0.4)	(0.6, 0.2, 0.2)	(0.5, 0.2, 0.3)
x_3	(0.3, 0.4, 0.3)	(0.8, 0.1, 0.1)	(0.7, 0.1, 0.2)
x_4	(0.8, 0.1, 0.1)	(0.7, 0.2, 0.1)	(0.9, 0.1, 0.0)
x_5	(0.6, 0.3, 0.1)	(0.8, 0.2, 0.0)	(0.7, 0.2, 0.1)

Table 4
Complex intuitionistic fuzzy decision matrix R.

	G_1	G_2	G_3
x_1	(0.7752, 0.1000, 0.1248)	(0.8688, 0.1312, 0.0000)	(0.7286, 0.1210, 0.1504)
x_2	(0.6158, 0.1738, 0.2104)	(0.5681, 0.1712, 0.2607)	(0.5736, 0.2211, 0.2053)
x_3	(0.3873, 0.4319, 0.1808)	(0.7483, 0.1738, 0.0778)	(0.7065, 0.1312, 0.1623)
x_4	(0.8485, 0.1000, 0.0515)	(0.7548, 0.1712, 0.0740)	(0.8152, 0.1848, 0.0000)
x_5	(0.5681, 0.2338, 0.1981)	(0.7339, 0.2314, 0.0347)	(0.6249, 0.1809, 0.1942)

Step 2. Determine the IFIS α^+, and the alternatives x_i $(i = 1, 2, 3, 4, 5)$ by

$$\alpha^+ = ((0.8485, 0.1000, 0.0515), (0.8688, 0.1312, 0.0000), (0.8152, 0.1848, 0.0000))^T,$$
$$x_1 = ((0.7752, 0.1000, 0.1248), (0.8688, 0.1312, 0.0000), (0.7286, 0.1210, 0.1504))^T,$$
$$x_2 = ((0.6158, 0.1738, 0.2104), (0.5681, 0.1712, 0.2607), (0.5736, 0.2211, 0.2053))^T,$$
$$x_3 = ((0.3873, 0.4319, 0.1808), (0.7483, 0.1738, 0.0778), (0.7065, 0.1312, 0.1623))^T,$$
$$x_4 = ((0.8485, 0.1000, 0.0515), (0.7548, 0.1712, 0.0740), (0.8152, 0.1848, 0.0000))^T,$$
$$x_5 = ((0.5681, 0.2338, 0.1981), (0.7339, 0.2314, 0.0347), (0.6249, 0.1809, 0.1942))^T.$$

Step 3. Utilize (13)–(21) to compute the values $S_h(x_i)$, $R_h(x_i)$ and $Q_h(x_i)$ $(h = d_1, d_2, d_3)$ for each alternative x_i $(i = 1, 2, 3, 4, 5)$, respectively, and rank the alternatives $x_i (i = 1, 2, 3, 4, 5)$ by sorting S_h, R_h and Q_h in an decreasing order (see Table 5).

Step 4. (a) In the case of d_1, when $0 \leq \upsilon < 0.8151$, the ranked order of all alternatives is $x_1 \succ x_4 \succ x_5 \succ x_2 \succ x_3$ and $\{x_1, x_4\}$ is the set of compromise solutions; when $\upsilon = 0.8151$, the ranked order of all alternatives is $x_1 \sim x_4 \succ x_5 \succ x_2 \succ x_3$ and $\{x_1, x_4\}$ is the set of compromise solutions; when $0.8151 < \upsilon \leq 1$, the ranked order of all alternatives is $x_4 \succ x_1 \succ x_5 \succ x_2 \succ x_3$ and $\{x_1, x_4\}$ is the set of compromise solutions.

(b) In the case of d_2, when $0 \leq \upsilon < 0.0516$, the ranked order of all alternatives is $x_1 \succ x_4 \succ x_5 \succ x_2 \succ x_3$ and $\{x_1, x_4\}$ is the set of compromise solutions; when $\upsilon = 0.0516$, the ranked order of all alternatives is $x_1 \sim x_4 \succ x_5 \succ x_2 \succ x_3$ and $\{x_1, x_4\}$ is the set of compromise solutions; when $0.0516 < \upsilon < 0.6056$, the ranked order of all alternatives is $x_4 \succ x_1 \succ x_5 \succ x_2 \succ x_3$ and $\{x_1, x_4\}$ is the set of compromise solutions; when $\upsilon = 0.6056$, the ranked order of all alternatives is $x_4 \succ x_1 \succ x_5 \succ x_2 \sim x_3$ and $\{x_1, x_4\}$ is the set of compromise solutions; when $0.6056 < \upsilon \leq 1$, the ranked order of all alternatives is $x_4 \succ x_1 \succ x_5 \succ x_3 \succ x_2$ and $\{x_1, x_4\}$ is the set of compromise solutions.

Table 5
Decision results obtained from the proposed Procedure I.

(a) The values $S_{d_1}(x_i)$, $R_{d_1}(x_i)$ and $Q_{d_1}(x_i)$ and preference ranking order obtained from d_1			
	S_{d_1}	R_{d_1}	Q_{d_1}
x_1	0.0336	0.0226	0.0193v
x_2	0.1558	0.0681	$0.4720 + 0.3895v$
x_3	0.1759	0.1190	1.0000
x_4	0.0308	0.0308	$0.0851(1 - v)$
x_5	0.1383	0.0621	$0.4098 + 0.3311v$
Ranking order	$x_4 \succ x_1 \succ x_5 \succ x_2 \succ x_3$	$x_1 \succ x_4 \succ x_5 \succ x_2 \succ x_3$	$x_1 \succ x_4 \succ x_5 \succ x_2 \succ x_3$ $(0 \le v < 0.8151)$ $x_1 \sim x_4 \succ x_5 \succ x_2 \succ x_3$ $(v = 0.8151)$ $x_4 \succ x_1 \succ x_5 \succ x_2 \succ x_3$ $(0.8151 < v \le 1)$
(b) The values $S_{d_2}(x_i)$, $R_{d_2}(x_i)$ and $Q_{d_2}(x_i)$ and preference ranking order obtained from d_2			
	S_{d_2}	R_{d_2}	Q_{d_2}
x_1	0.0671	0.0451	$0.0991v$
x_2	0.2626	0.1203	$0.8060 + 0.1940v$
x_3	0.2352	0.1384	$1 - 0.1263v$
x_4	0.0456	0.0456	$0.0054(1 - v)$
x_5	0.1963	0.0841	$0.4180 + 0.2765v$
Ranking order	$x_4 \succ x_1 \succ x_5 \succ x_3 \succ x_2$	$x_1 \succ x_4 \succ x_5 \succ x_2 \succ x_3$	$x_1 \succ x_4 \succ x_5 \succ x_2 \succ x_3$ $(0 \le v < 0.0516)$ $x_1 \sim x_4 \succ x_5 \succ x_2 \succ x_3$ $(v = 0.0516)$ $x_4 \succ x_1 \succ x_5 \succ x_2 \succ x_3$ $(0.0516 < v < 0.6056)$ $x_4 \succ x_1 \succ x_5 \succ x_2 \sim x_3$ $(v = 0.6056)$ $x_4 \succ x_1 \succ x_5 \succ x_3 \succ x_2$ $(0.6056 < v \le 1)$
(c) The values $S_{d_3}(x_i)$, $R_{d_3}(x_i)$ and $Q_{d_3}(x_i)$ and preference ranking order obtained from d_h			
	S_{d_3}	R_{d_3}	Q_{d_3}
x_1	0.0480	0.0260	$0.0111v$
x_2	0.2626	0.1203	$0.8390 + 0.1610v$
x_3	0.2192	0.1384	$1 - 0.2v$
x_4	0.0456	0.0456	$0.1744(1 - v)$
x_5	0.1952	0.0841	$0.5169 + 0.1725v$
Ranking order	$x_4 \succ x_1 \succ x_5 \succ x_3 \succ x_2$	$x_1 \succ x_4 \succ x_5 \succ x_2 \succ x_3$	$x_1 \succ x_4 \succ x_5 \succ x_2 \succ x_3$ $(0 \le v < 0.4460)$ $x_1 \succ x_4 \succ x_5 \succ x_2 \sim x_3$ $(v = 0.4460)$ $x_1 \succ x_4 \succ x_5 \succ x_3 \succ x_2$ $(0.4460 < v < 0.9402)$ $x_1 \sim x_4 \succ x_5 \succ x_3 \succ x_2$ $(v = 0.9402)$ $x_4 \succ x_1 \succ x_5 \succ x_3 \succ x_2$ $(0.9402 < v \le 1)$

(c) In the case of d_3, when $0 \le v < 0.4460$, the ranked order of all alternatives is $x_1 \succ x_4 \succ x_5 \succ x_2 \succ x_3$ and $\{x_1, x_4\}$ is the set of compromise solutions; when $v = 0.4460$, the ranked order of all alternatives is $x_1 \succ x_4 \succ x_5 \succ x_2 \sim x_3$ and $\{x_1, x_4\}$ is the set of compromise solutions; when $0.4460 < v < 0.9402$, the ranked order of all alternatives is $x_4 \succ x_1 \succ x_5 \succ x_3 \succ x_2$ and $\{x_1, x_4\}$ is the set of compromise solutions; when $v = 0.9402$, the ranked order of all alternatives is $x_4 \sim x_1 \succ x_5 \succ x_3 \sim x_2$ and $\{x_1, x_4\}$ is the set of compromise solutions; when $0.9402 < v \le 1$, the ranked order of all alternatives is $x_4 \succ x_1 \succ x_5 \succ x_3 \succ x_2$ and $\{x_1, x_4\}$ is the set of compromise solutions.

From the results above, we can see that the ranking orders of all alternatives x_i are different because of the values of v and the distance measures d_1, d_2 and d_3; while the set of compromise solutions are the same.

If the committee evaluates the performance of five faculty candidates $x_i (i = 1, 2, 3, 4, 5)$ in the years t_k $(k = 1, 2, 3)$ according to attributes $G_j (j = 1, 2, 3)$, and constructs, respectively, the uncertain intuitionistic fuzzy decision matrices $\tilde{R}(t_k) (k = 1, 2, 3)$ as listed in Tables 6–8. In such case, we can utilize the proposed Procedure II presented in Section 5 to prioritize these faculty candidates. To do so, we first the UDIFWG operator to aggregate all the uncertain intuitionistic fuzzy decision matrices $\tilde{R}(t_k)$ into a complex uncertain intuitionistic fuzzy decision matrix $\tilde{R}$ (see Table 9): and then denote the

Table 6
Uncertain intuitionistic fuzzy decision matrix $\tilde{R}(t_1)$.

	G_1	G_2	G_3
x_1	([0.8, 0.9], [0.0, 0.1], [0.0, 0.2])	([0.8, 0.9], [0.0, 0.1], [0.1, 0.2])	([0.6, 0.8], [0.0, 0.2], [0.0, 0.4])
x_2	([0.6, 0.7], [0.2, 0.3], [0.0, 0.2])	([0.5, 0.7], [0.2, 0.3], [0.0, 0.3])	([0.5, 0.6], [0.2, 0.3], [0.1, 0.3])
x_3	([0.4, 0.5], [0.2, 0.4], [0.1, 0.4])	([0.5, 0.6], [0.2, 0.3], [0.1, 0.3])	([0.4, 0.6], [0.1, 0.2], [0.2, 0.5])
x_4	([0.8, 0.9], [0.0, 0.1], [0.0, 0.2])	([0.6, 0.8], [0.0, 0.1], [0.1, 0.4])	([0.6, 0.7], [0.1, 0.2], [0.1, 0.3])
x_5	([0.5, 0.7], [0.1, 0.3], [0.0, 0.4])	([0.7, 0.8], [0.1, 0.2], [0.0, 0.2])	([0.4, 0.5], [0.2, 0.4], [0.1, 0.4])

Table 7
Uncertain intuitionistic fuzzy decision matrix $\tilde{R}(t_2)$.

	G_1	G_2	G_3
x_1	([0.7, 0.8], [0.1, 0.2], [0.0, 0.2])	([0.8, 0.9], [0.0, 0.1], [0.0, 0.2])	([0.7, 0.9], [0.0, 0.1], [0.0, 0.3])
x_2	([0.5, 0.7], [0.1, 0.2], [0.1, 0.4])	([0.6, 0.7], [0.1, 0.3], [0.0, 0.3])	([0.6, 0.8], [0.1, 0.2], [0.0, 0.3])
x_3	([0.3, 0.5], [0.1, 0.3], [0.2, 0.6])	([0.4, 0.5], [0.1, 0.3], [0.2, 0.5])	([0.3, 0.6], [0.3, 0.4], [0.0, 0.4])
x_4	([0.6, 0.7], [0.1, 0.2], [0.1, 0.3])	([0.7, 0.8], [0.1, 0.2], [0.0, 0.2])	([0.5, 0.7], [0.1, 0.3], [0.0, 0.4])
x_5	([0.5, 0.7], [0.2, 0.3], [0.0, 0.3])	([0.5, 0.7], [0.1, 0.3], [0.0, 0.4])	([0.4, 0.6], [0.2, 0.3], [0.1, 0.4])

Table 8
Uncertain intuitionistic fuzzy decision matrix $\tilde{R}(t_3)$.

	G_1	G_2	G_3
x_1	([0.6, 0.7], [0.1, 0.3], [0.0, 0.3])	([0.7, 0.9], [0.0, 0.1], [0.0, 0.3])	([0.6, 0.8], [0.0, 0.1], [0.1, 0.4])
x_2	([0.4, 0.6], [0.1, 0.2], [0.2, 0.5])	([0.5, 0.7], [0.1, 0.2], [0.1, 0.4])	([0.6, 0.7], [0.1, 0.3], [0.0, 0.3])
x_3	([0.2, 0.4], [0.2, 0.3], [0.3, 0.6])	([0.3, 0.6], [0.2, 0.3], [0.1, 0.5])	([0.4, 0.6], [0.2, 0.4], [0.0, 0.4])
x_4	([0.7, 0.8], [0.0, 0.1], [0.1, 0.3])	([0.8, 0.9], [0.0, 0.1], [0.0, 0.2])	([0.7, 0.8], [0.1, 0.2], [0.0, 0.2])
x_5	([0.5, 0.6], [0.2, 0.3], [0.1, 0.3])	([0.8, 0.9], [0.0, 0.1], [0.0, 0.2])	([0.6, 0.7], [0.2, 0.3], [0.0, 0.2])

Table 9
Complex uncertain intuitionistic fuzzy decision matrix $\tilde{R}$.

	G_1	G_2
x_1	([0.67, 0.77], [0.08, 0.23], [0.00, 0.25])	([0.75, 0.90], [0.00, 0.10], [0.00, 0.25])
x_2	([0.46, 0.65], [0.12, 0.22], [0.13, 0.42])	([0.53, 0.70], [0.12, 0.25], [0.05, 0.35])
x_3	([0.26, 0.45], [0.17, 0.32], [0.23, 0.57])	([0.36, 0.57], [0.17, 0.30], [0.13, 0.47])
x_4	([0.69, 0.79], [0.03, 0.13], [0.08, 0.28])	([0.73, 0.85], [0.03, 0.13], [0.02, 0.24])
x_5	([0.50, 0.65], [0.18, 0.30], [0.05, 0.32])	([0.68, 0.82], [0.05, 0.18], [0.00, 0.27])

	G_3
x_1	([0.63, 0.83], [0.00, 0.12], [0.05, 0.37])
x_2	([0.58, 0.71], [0.12, 0.27], [0.02, 0.30])
x_3	([0.37, 0.60], [0.21, 0.36], [0.04, 0.42])
x_4	([0.61, 0.75], [0.10, 0.23], [0.02, 0.29])
x_5	([0.49, 0.62], [0.20, 0.32], [0.05, 0.31])

UIFIS $\tilde{\alpha}^+$, and the alternatives $x_i (i = 1, 2, 3, 4, 5)$ by

$$\begin{aligned}
\alpha^+ &= (([0.69, 0.79], [0.03, 0.13], [0.08, 0.28]), ([0.75, 0.90], [0.00, 0.10], [0.00, 0.25]),\\
&\quad ([0.63, 0.83], [0.00, 0.12], [0.05, 0.37]))^T,\\
x_1 &= (([0.67, 0.77], [0.08, 0.23], [0.00, 0.25]), ([0.75, 0.90], [0.00, 0.10], [0.00, 0.25]),\\
&\quad ([0.63, 0.83], [0.00, 0.12], [0.05, 0.37]))^T,\\
x_2 &= (([0.46, 0.65], [0.12, 0.22], [0.13, 0.42]), ([0.53, 0.70], [0.12, 0.25], [0.05, 0.35]),\\
&\quad ([0.58, 0.71], [0.12, 0.27], [0.02, 0.30]))^T,\\
x_3 &= (([0.26, 0.45], [0.17, 0.32], [0.23, 0.57]), [0.36, 0.57], [0.17, 0.30], [013, 0.47]),\\
&\quad ([0.37, 0.60], [0.21, 0.36], [0.04, 0.42])^T,\\
x_4 &= (([0.69, 0.79], [0.03, 0.13], [0.08, 0.28]), ([0.73, 0.85], [0.03, 0.13], [0.02, 0.24]),\\
&\quad ([0.61, 0.75], [0.10, 0.23], [0.02, 0.29]))^T,\\
x_5 &= (([0.50, 0.65], [0.18, 0.30], [0.05, 0.32]), ([0.68, 0.82], [0.05, 0.18], [0.00, 0.27]),\\
&\quad ([0.49, 0.62], [0.20, 0.32], [0.05, 0.31]))^T.
\end{aligned}$$

Table 10
Decision results obtained from the proposed Procedure II.

(a) The values $\tilde{S}_{d_1}(x_i)$, $\tilde{R}_{d_1}(x_i)$ and $\tilde{Q}_{d_1}(x_i)$ and preference ranking order obtained from d_1			
	$\tilde{S}_{d_1}$	$\tilde{R}_{d_1}$	$\tilde{Q}_{d_1}$
x_1	0.0330	0.0143	0.0271υ
x_2	0.1470	0.0690	$0.5776 - 0.1086\upsilon$
x_3	0.2840	0.1090	1
x_4	0.0370	0.0233	$0.0950 - 0.0524\upsilon$
x_5	0.0260	0.0563	$0.4435(1-\upsilon)$
Ranking order	$x_5 \succ x_1 \succ x_4 \succ x_2 \succ x_3$	$x_1 \succ x_4 \succ x_5 \succ x_2 \succ x_3$	$x_1 \succ x_4 \succ x_5 \succ x_2 \succ x_3$ $(0 \le \upsilon < 0.8911)$ $x_1 \succ x_4 \sim x_5 \sim x_2 \succ x_3$ $(\upsilon = 0.8911)$ $x_1 \succ x_5 \succ x_4 \succ x_2 \succ x_3$ $(0.8911 < \upsilon < 0.9424)$ $x_1 \sim x_5 \succ x_4 \succ x_2 \succ x_3$ $(\upsilon = 0.9424)$ $x_5 \succ x_1 \succ x_4 \succ x_2 \succ x_3$ $(0.9424 < \upsilon \le 1)$
(b) The values $\tilde{S}_{d_2}(x_i)$, $\tilde{R}_{d_2}(x_i)$ and $\tilde{Q}_{d_2}(x_i)$ and preference ranking order obtained from d_2			
	$\tilde{S}_{d_2}$	$\tilde{R}_{d_2}$	$\tilde{Q}_{d_2}$
x_1	0.0225	0.0225	0
x_2	0.1800	0.0840	$0.5062 - 0.0014\upsilon$
x_3	0.3345	0.1440	1
x_4	0.0475	0.0315	$0.0741 + 0.0061\upsilon$
x_5	0.1448	0.0608	$0.3152 + 0.0768\upsilon$
Ranking order	$x_1 \succ x_4 \succ x_5 \succ x_2 \succ x_3$	$x_1 \succ x_4 \succ x_2 \succ x_5 \succ x_3$	$x_1 \succ x_4 \succ x_5 \succ x_2 \succ x_3$
(c) The values $\tilde{S}_{d_3}(x_i)$, $\tilde{R}_{d_3}(x_i)$ and $\tilde{Q}_{d_3}(x_i)$ and preference ranking order obtained from d_h			
	$\tilde{S}_{d_3}$	$\tilde{R}_{d_3}$	$\tilde{Q}_{d_3}$
x_1	0.0125	0.0125	0
x_2	0.1215	0.0643	$0.4739 + 0.0557\upsilon$
x_3	0.2183	0.1218	1
x_4	0.0223	0.0143	$0.0165 + 0.0312\upsilon$
x_5	0.1070	0.0603	$0.4373 + 0.0219\upsilon$
Ranking order	$x_1 \succ x_4 \succ x_5 \succ x_2 \succ x_3$	$x_1 \succ x_4 \succ x_5 \succ x_2 \succ x_3$	$x_1 \succ x_4 \succ x_5 \succ x_2 \succ x_3$

By (34)–(42), we calculate the values $\tilde{S}_h(x_i)$, $\tilde{R}_h(x_i)$ and $\tilde{Q}_h(x_i)$ $(h = d_1, d_2, d_3)$ for each alternative x_i $(i = 1, 2, 3, 4, 5)$, respectively, and rank the alternatives $x_i (i = 1, 2, 3, 4, 5)$ by sorting $\tilde{S}_h$, $\tilde{R}_h$ and $\tilde{Q}_h$ in decreasing order (see Table 10).

(a) In the case of d_1, when $0 \le \upsilon < 0.8911$, the ranked order of all alternatives is $x_1 \succ x_4 \succ x_5 \succ x_2 \succ x_3$ and $\{x_1, x_4\}$ is the set of compromise solutions if $0 \le \upsilon \le 0.4363$, and $\{x_1, x_4, x_5\}$ is the set of compromise solutions if $0.4363 < \upsilon < 0.8911$; when $\upsilon = 0.8911$, the ranked order of all alternatives is $x_1 \succ x_4 \sim x_5 \succ x_2 \succ x_3$ and $\{x_1, x_4, x_5\}$ is the set of compromise solutions; when $0.8911 < \upsilon < 0.9424$, the ranked order of all alternatives is $x_1 \succ x_5 \succ x_4 \succ x_2 \succ x_3$ and $\{x_1, x_4, x_5\}$ is the set of compromise solutions; when $\upsilon = 0.9424$, the ranked order of all alternatives is $x_1 \sim x_5 \succ x_4 \succ x_2 \succ x_3$ and $\{x_1, x_4, x_5\}$ is the set of compromise solutions; when $0.9424 < \upsilon \le 1$, the ranked order of all alternatives is $x_5 \succ x_1 \succ x_4 \succ x_2 \succ x_3$ and $\{x_1, x_4, x_5\}$ is the set of compromise solutions.

(b) In the case of d_2, the ranked order of all alternatives is $x_1 \succ x_4 \succ x_5 \succ x_2 \succ x_3$ and $\{x_1, x_4\}$ is the set of compromise solutions.

(c) In the case of d_3, the ranked order of all alternatives is $x_1 \succ x_4 \succ x_5 \succ x_2 \succ x_3$ and $\{x_1, x_4\}$ is the set of compromise solutions.

From the results above, we can see that, in the case of d_1, the ranking orders of all alternatives x_i are different because of the values of υ and thus the set of compromise solutions are also different; while, in the cases of d_2 and d_3, the ranking orders of all alternatives are the same regardless of the values of υ, and thus the set of compromise solutions are the same.

7. Conclusion

In this paper, we have studied on the dynamic intuitionistic fuzzy multiple attribute decision making (DIF-MADM) problems, which occur in many decision areas, such as multi-period investment decision making, medical diagnosis, personal dynamic examination and military system efficiency dynamic evaluation. Some aggregation operators such as the dynamic intuitionistic fuzzy weighted geometric (DIFWG) operator and uncertain dynamic intuitionistic fuzzy weighted geometric (UDIFWG) operator have been proposed to aggregate dynamic or uncertain dynamic intuitionistic fuzzy information. Based on the DIFWG operator and UDIFWG operators respectively, we have developed two approaches for solving the DIF-MADM problems where all the attribute values are expressed in intuitionistic fuzzy numbers or interval-valued intuitionistic fuzzy numbers. In the proposed approaches, we have extended the VIKOR method to intuitionistic fuzzy environment, and used the extended VIKOR method to rank and select the optimal alternative. To verify the effectiveness and practicality of the developed approaches, we have applied them to evaluate university faculty for tenure and promotion.

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