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A multi-criteria interval-valued intuitionistic fuzzy group decision making with Choquet integral-based TOPSIS

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ABSTRACT

An extension of TOPSIS, a multi-criteria interval-valued intuitionistic fuzzy decision making technique, to a group decision environment is investigated, where inter-dependent or interactive characteristics among criteria and preference of decision makers are taken into account. To get a broad view of the techniques used, first, some operational laws on interval-valued intuitionistic fuzzy values are introduced. Based on these operational laws, a generalized interval-valued intuitionistic fuzzy geometric aggregation operator is proposed which is used to aggregate decision makers' opinions in group decision making process. In addition, some of its properties are discussed. Then Choquet integral-based Hamming distance between interval-valued intuitionistic fuzzy values is defined. Combining the interval-valued intuitionistic fuzzy geometric aggregation operator with Choquet integral-based Hamming distance, an extension of TOPSIS method is developed to deal with a multi-criteria interval-valued intuitionistic fuzzy group decision making problems. Finally, an illustrative example is used to illustrate the developed procedures.

1. Introduction

TOPSIS (Technique for Order Preference by Similarity to Ideal Solution), developed by Hwang and Yoon (1981), is a classical approach to multi-attribute or multi-criteria decision making (MADM/MCDM) problems. It is a practical and useful technique for ranking and selection of a number of externally determined alternatives through distance measures. The basic principle is that the chosen alternative should have the shortest distance from the positive-ideal solution and the farthest distance from the negative-ideal solution. There exists a large amount of literature involving TOPSIS theory and applications. In the TOPSIS, the performance ratings and the weights of the criteria are given as crisp values. Under many conditions, crisp values are inadequate to model real-world situations because human judgment and preference are often ambiguous and cannot be estimated with exact numerical values. To resolve the ambiguity frequently arising in information from human judgment and preference, fuzzy set theory (Zadeh, 1965) has been successfully used to handle imprecision (or uncertainty) in decision making problems. Since fuzzy numbers were applied to establish a prototype fuzzy TOPSIS (Chen & Hwang, 1992; Negi, 1989), many works on fuzzy TOPSIS have been investigated (Chen, 2000; Chu & Lin, 2009; Jahanshahloo, Hosseinzadeh Lotfi, & Izadikhah, 2006; Kuo, Tzeng, & Huang, 2007; Mahdavi, Mahdavi-Amiri, Heidarzade, & Nourifar, 2008; Wang & Chang, 2007; Wang & Elhag, 2006; Wang & Lee, 2007, 2009; Yeh & Deng, 2004; Yeh, Deng, & Chang, 2000). As an extension of Zadeh's fuzzy set whose basic component is only a membership function, Atanassov (1986) introduced the intuitionistic fuzzy sets (IFS), characterized by a membership function and a non-membership function, Accordingly, IFS has been proven to be a very suitable tool to be used to describe the imprecise or uncertain decision information. A lot of work has been done to develop and enrich the IFS theory (Atanassov, 1999; Bustince, Herrera, & Montero, 2007). As a generalization of the fuzzy sets, IFS has received more and more attention and has been applied to the field of decision making. And fuzzy TOPSIS has been extended to IFS (Ashtiani, Haghighirad, Makui, & Montazer, 2009; Boran, Gen, Kurt, & Akay, 2009; Chen & Tsao, 2008; Li, Wang, Liu, & Shan, 2008). Later, Atanassov and Gargov (1989) introduced the concept of interval-valued intuitionistic fuzzy sets (IVIFS) as a further generalization of that of IFS. The fundamental characteristic of the IVIFS is that the values of its membership function and non-membership function are intervals rather than exact numbers. Atanassov (1994) defined some operational laws of the IVIFS. Recently, Tan and Zhang (2006) presented a novel method for multiple attribute decision making based on IVIFS and TOPSIS method in uncertain environments. Xu (2007) developed some geometric aggregation operators, such as the interval-valued intuitionistic fuzzy weighted geometric averaging (IIFWGA) operator and the interval-valued intuitionistic fuzzy ordered weighted geometric averaging (IIFOWGA) operator and gave an application of the IIFWGA and IIFOWGA operators to multiple attribute group decision making with interval-valued intuitionistic fuzzy information. Wei (2009) applied IIFWGA aggregation functions to dealing with dynamic multiple attribute decision making

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where all the attribute values are expressed in intuitionistic fuzzy numbers or interval-valued intuitionistic fuzzy numbers.

However, these aggregation process are based on the assumption that the criteria (attribute) or preferences of decision makers are independent, and the aggregation operators are linear operators based on additive measures, which is characterized by an independence axiom (Keeney & Raiffa, 1976; Wakker, 1999). For real decision making problems, there is a phenomenon that there exists some degree of inter-dependent or interactive characteristics between criteria (Grabisch, 1995; Grabisch, Murofushi, & Sugeno, 2000). And for a decision problem, decision makers invited usually come from same or similar fields. They have similar knowledge, social status and preference. Decision makers' subjective preference always shows non-linearity. Independence phenomena among these criteria and mutual preferential independence of decision makers are violated. In 1974, Sugeno (1974) introduced the concept of non-additive measure (fuzzy measure), which only make a monotonicity instead of additivity property. It is a most effective tool to modeling interaction phenomena (Grabisch, 1996; Ishii & Sugeno, 1985; Kojadinovic, 2002; Roubens, 1996) and deal with decision problems (Grabisch, 1995, 1997; Grabisch et al., 2000; Onisawa, Sugeno, Nishiwaki, Kawai, & Harima, 1986). A review on analyzing decision maker behavior using fuzzy measure theory can be seen in Liginlal and Ow (2006). In group decision making problems, aggregation of decision makers' opinions is very important to appropriately perform evaluation process. To overcome this limitation of above aggregation operator, in this paper, based on fuzzy measure we first shall develop a generalized interval-valued intuitionistic fuzzy geometric aggregation operator for aggregating all individual decision makers' opinions under interval-valued intuitionistic fuzzy group decision making environment. Combining this operator with TOPSIS on Choquet integral-based Hamming distance, a multi-criteria interval-valued intuitionistic fuzzy group decision making is investigated, where interactions phenomena among the decision making problem are considered.

In order to do this, the paper is organized as follows: In Section 2, we review fuzzy measure. In Section 3, we introduce interval-valued intuitionistic fuzzy set and some operational laws on interval-valued intuitionistic fuzzy values, In Section 4, based on these operational laws, a generalized interval-valued intuitionistic fuzzy geometric aggregation operator is proposed, and some of its properties are investigated. In Section 5, according to definition of Choquet integral, we invoke the well-known Hamming distance to define the Choquet integral-based Hamming distance between any two interval-valued intuitionistic fuzzy sets. Combining the generalized interval-valued intuitionistic fuzzy geometric aggregation operator with Choquet integral-based Hamming distance, an extension of TOPSIS is developed to deal with a multi-criteria interval-valued intuitionistic fuzzy group decision making problems where inter-dependent or interactive characteristics among criteria and preference of decision makers are taken into account. In Section 6, an example is given to illustrate the concrete application of the method and to demonstrate its feasibility and practicality. Conclusions are made in Section 7.

2. Fuzzy measure

For traditional additive aggregation operators, such as the weighted arithmetic mean or OWA (Yager, 1988) operator, each criteria $i \in N$ (N denotes a criteria set) is given a weight $w_i \in [0,1]$ representing the importance of this criteria during the decision process, and the sun of all w_i $(i = 1,2,\ldots,n)$ amount to one. But it does not define a weight on each combination of criteria. In real decision problems, since there are often inter-dependent or interactive phenomena among criteria, the overall importance of a criterion $i \in N$ is not solely determined by itself i, but also by all other criteria T, $i \in T$. Suppose that $w(i)$ denotes the importance degree of i, we may have $w(i) = 0$, suggesting that element is unimportant, but it may happen that for many subsets $T \subseteq N$, $w(T \cup i)$ is much greater than $w(T)$, suggesting that i is actually an important element in the decision. In 1974, Sugeno (1974) introduced the concept of fuzzy measure (non-additive measure), which only make a monotonicity instead of additivity property. For real decision making problems, fuzzy measure define a weight on not only each criteria but also each combination of criteria, and the sun of every w_i $(i = 1,2,\ldots,n)$ does not equal to one. Thus it is used as a powerful tool for describing the interaction among the criteria in a set.

Definition 1. Let $X = \{x_1,x_2,\ldots,x_n\}$ be a universe of discourse, $P(X)$ be the power set of X. A fuzzy measure on X is a set function $\mu : P(X) \rightarrow [0,1]$, satisfying the following conditions:

(1) $\mu(\phi) = 0$, $\mu(X) = 1$.
(2) If $A, B \in P(X)$ and $A \subseteq B$ then $\mu(A) \leqslant \mu(B)$.

If the universal set X is infinite, it is necessary to add an extra axiom of continuity (Wang & Klir, 1992). However, in actual practice, it is enough to consider the finite universal set. $\mu(S)$ can be viewed as the grade of subjective importance of decision criteria set S. Thus, in addition to the usual weights on criteria taken separately, weights on any combination of criteria are also defined. This makes possible the representation of interaction between criteria. Let $E_j = \{x_j,x_{j+1},\ldots,x_n\}$ $(1 \leqslant j \leqslant n)$ be a criteria set. The interaction among the criteria in E_j can be described by employing $\mu(E_j)$ to express the degree of importance of E_j. That is, the degree of importance of E_j is evaluated by simultaneously considering x_j, $x_{j+1},\ldots,$ and x_n. Hence, μ can be called an importance measure (Wang, Wang, & Klir, 1998), and $\mu(E_j)$ can be also employed to express the discriminatory power of E_j. Intuitively, we could say the following about any a pair of criteria sets $A, B \in P(X)$, $A \cap B = \phi$: A and B are considered to be without interaction (or to be independent) if $\mu(A \cup B) = \mu(A) + \mu(B)$, which is called an additive measure. A and B exhibit a positive synergetic interaction between them (or are complementary) if $\mu(A \cup B) > \mu(A) + \mu(B)$, which is called a super-additive measure. A and B exhibit a negative synergetic interaction between them (or are redundant or substitutive) if $\mu(A \cup B) < \mu(A) + \mu(B)$, which is called a sub-additive measure.

In order to determine such fuzzy measure, we generally need to find $2^n - 2$ values for n criteria, only values $\mu(\phi)$ and μ (X) are always equal to 0 and 1, respectively. So the evaluation model obtained becomes quite complex, and the structure is difficult to grasp. To avoid the problems with computational complexity and practical estimations, λ-fuzzy measure g, a special kind of fuzzy measure, was proposed by Sugeno (1974), which satisfies the following additional property:

$$g(A \cup B) = g(A) + g(B) + \lambda g(A)g(B), \tag{1}$$

where $\lambda > -1$ for all A, $B \in P(X)$ and $A \cap B = \phi$. However, there are several methods for the determination of the fuzzy measure. For instance, linear methods (Marichal & Roubens, 1998), quadratic methods (Grabisch, 1996; Grabisch & Nicolas, 1994), heuristic-based methods (Grabisch, 1995) and genetic algorithms (Wang et al., 1998) are available in the literature.

In Eq. (1), $\lambda = 0$ indicates that the λ-fuzzy measure g is additive measure. $\lambda \neq 0$ indicates that the λ-fuzzy measure g is non-additive and there is interaction between A and B. If $\lambda > 0$, then $g(A \cup B) > g(A) + g(B)$, which implies that g is a super-additive measure. If $\lambda < 0$, then $g(A \cup B) < g(A) + g(B)$, which implies that g is a sub -additive measure. By parameter λ the interaction between criteria can be represented.

If X is a finite set, then $\cup_{i=1}^{n} x_i = X$. The λ-fuzzy measure g satisfies following Eq. (2)

$$g(X) = g(\bigcup_{i=1}^{n} x_i) = \begin{cases} \frac{1}{\lambda}\left(\prod_{i=1}^{n}[1+\lambda g(x_i)] - 1\right) & \text{if } \lambda \neq 0, \\ \sum_{i=1}^{n} g(x_i) & \text{if } \lambda = 0, \end{cases} \tag{2}$$

where $x_i \cap x_j = \phi$ for all $i,j = 1,2,\ldots,n$ and $i \neq j$. It can be noted that $g(x_i)$ for a subset with a single element x_i is called a fuzzy density, and can be denoted as $g_i = g(x_i)$.

Especially for every subset $A \in P(X)$, we have

$$g(A) = \begin{cases} \frac{1}{\lambda}\left(\prod_{i \in A}[1+\lambda g(i)] - 1\right) & \text{if } \lambda \neq 0, \\ \sum_{i \in A} g(i) & \text{if } \lambda = 0. \end{cases} \tag{3}$$

Based on Eq. (2), the value λ of can be uniquely determined from $g(X) = 1$, which is equivalent to solving

$$\lambda + 1 = \prod_{i=1}^{n}(1 + \lambda g_i). \tag{4}$$

It should be noted that λ can be uniquely determined by $g(X) = 1$.

3. Interval-valued intuitionistic fuzzy sets

Let X be a universe of discourse, a fuzzy set in X is an expression A given by

$$A = \{\langle x, t_A(x)\rangle | x \in X\},$$

where $t_A : X \to [0,1]$ is a membership function which characterizes the degree of membership of the element x to the set A. The main characteristic of fuzzy sets is that: the membership function assigns to each element x in a universe of discourse X a membership degree in interval $[0,1]$ and the non-membership degree equals one minus the membership degree, i.e., this single membership degree combines the evidence for x and the evidence against x, without indicating how much there is of each. The single membership value tells us nothing about the lack of knowledge. In real applications, however, the information of an object corresponding to a fuzzy concept may be incomplete, i.e., the sum of the membership degree and the non-membership degree of an element in a universe corresponding to a fuzzy concept may be less than one. In fuzzy set theory, there is no means to incorporate the lack of knowledge with the membership degrees. In 1986, Atanassov (1986) generalized the concept of fuzzy set, and defined the concept of intuitionistic fuzzy set as follows.

Definition 2. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a universe of discourse, an intuitionistic fuzzy set in X is an expression A given by

$$A = \{\langle x, t_A(x_i), f_A(x_i)\rangle | x_i \in X\},$$

where $t_A : X \to [0,1]$, $f_A : X \to [0,1]$ with the condition: $0 \leqslant t_A(x_i) + f_A(x_i) \leqslant 1$, for all x_i in X. The numbers $t_A(x_i)$ and $f_A(x_i)$ represent the degree of membership and the degree of non-membership of the element x_i in the set A, respectively.

For each intuitionistic fuzzy set A in X, if

$$\pi_A(x) = 1 - t_A(x) - f_A(x), \quad \forall x \in X.$$

Then $\pi_A(x)$ is called the degree of indeterminacy of x to A. Especially, if

$$\pi_A(x) = 1 - t_A(x) - f_A(x) = 0, \quad \forall x \in X,$$

then the intuitionistic fuzzy set A is reduced to a fuzzy set.

Atanassov and Gargov (1989) further introduced the interval-valued intuitionistic fuzzy set (IVIFS), which is a generalization of the IFS. The fundamental characteristic of the IVIFS is that the values of its membership function and non-membership function are intervals rather than exact numbers.

Definition 3. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a universe of discourse, $D[0,1]$ be the set of all closed subintervals of the interval $[0,1]$. An interval-valued intuitionistic fuzzy set A in X is an expression given by

$$A = \{\langle x, t_A(x_i), f_A(x_i)\rangle | x_i \in X\}, \tag{5}$$

where $t_A : X \to D[0,1]$, $f_A : X \to D[0,1]$ with the condition $0 \leqslant \sup t_A(x_i) + \sup f_A(x_i) \leqslant 1$. The intervals $t_A(x_i)$ and $f_A(x_i)$ denote, respectively, the degree of belongingness and the degree of non-belongingness of the element x_i to the set A.

For any two intervals $[a,b]$ and $[c,d]$ with $b + d \leqslant 1$ belonging to $D[0,1]$, let $t_A(x) = [a,b]$, $f_A(x) = [c,d]$, so an interval-valued intuitionistic fuzzy set whose value is denoted by $A = \{\langle x, [a,b], [c,d] > | x \in X\}$. In this paper, we call $([a,b],[c,d])$ an interval-valued intuitionistic fuzzy value. For convenience, let Ω be the set of all interval-valued intuitionistic fuzzy values on X. Obviously, according to Definition 3, we know that $\tilde{a}^+ = ([1,1],[0,0])$ and $\tilde{a}^- = ([0,0],[1,1])$ are the largest and smallest interval-valued intuitionistic fuzzy values, respectively.

In the following, we define a distance measure between interval-valued intuitionistic fuzzy values.

Definition 4. Let $X = \{x_1, \ldots, x_n\}$ be a universe of discourse, $\tilde{a} = ([a_i, b_i], [c_i, d_i])$ and $\tilde{b} = ([a'_i, b'_i], [c'_i, d'_i])$ $(i = 1, 2, \ldots, n)$ be two interval-valued intuitionistic fuzzy values on X, then

$$d(\tilde{a}, \tilde{b}) = \frac{1}{4}\sum_{i=1}^{n} |a_i - a'_i| + |b_i - b'_i| + |c_i - c'_i| + |d_i - d'_i|,$$

is called the normalized Hamming distance between $\tilde{a}$ and $\tilde{b}$. If

$$d(\tilde{a}, \tilde{b}) = \frac{1}{4}\sum_{i=1}^{n} w_i(|a_i - a'_i| + |b_i - b'_i| + |c_i - c'_i| + |d_i - d'_i|),$$

where $w = (w_1, w_2, \ldots, w_n)$ is the weight vector of x_j such that $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, then $d(\tilde{a}, \tilde{b})$ is called the weighted Hamming distance between $\tilde{a}$ and $\tilde{b}$.

The following expressions are defined in Atanassov and Gargov (1989), Atanassov (1994) for any two interval-valued intuitionistic fuzzy values $\tilde{a}_1 = ([a_1, b_1], [c_1, d_1])$ and $\tilde{a}_2 = ([a_2, b_2], [c_2, d_2])$:

$$\begin{aligned} &(1)\ \tilde{a}_1 \leqslant \tilde{a}_2 \text{ iff } b_1 \leqslant b_2 \text{ and } a_1 \leqslant a_2 \text{ and } d_1 \geqslant d_2 \text{ and } c_1 \geqslant c_2. \\ &(2)\ \tilde{a}_1 = \tilde{a}_2 \text{ iff } b_1 = b_2 \text{ and } a_1 = a_2 \text{ and } d_1 = d_2 \text{ and } c_1 = c_2. \end{aligned} \tag{6}$$

However, Eq. (6) is not satisfied in some situations. So it cannot be used to compare intuitionistic fuzzy values. In the following, similar to IFS, we define a score function and an accuracy function of interval-valued intuitionistic fuzzy values for the comparison between two interval-valued intuitionistic fuzzy values.

Definition 5. Let $\tilde{a} = ([a,b],[c,d])$ be an interval-valued intuitionistic fuzzy values, if $S(\tilde{a}) = (a - c + b - d)/2$, then $S(\tilde{a})$ is called a score function of $\tilde{a}$, where $S(\tilde{a}) \in [-1,1]$.

Definition 6. Let $\tilde{a} = ([a,b],[c,d])$ be an interval-valued intuitionistic fuzzy values, if $H(\tilde{a}) = (a + b + c + d)/2$, then $H(\tilde{a})$ is called an accuracy function of $\tilde{a}$, where $H(\tilde{a}) \in [0,1]$.

As presented above, the score function S and the accuracy function H are, respectively, defined as the difference and the sum of the membership function $t_A(x)$ and the non-membership function

$f_A(x)$. In fact, the relation between the score function S and the accuracy function H is similar to the relation between mean and variance in statistics. Based on the score function S and the accuracy function H, in the following, Xu (2007) give an order relation between two interval-valued intuitionistic fuzzy values, which is defined as follows:

Definition 7. Let $\tilde{a}_1 = ([a_1, b_1], [c_1, d_1])$ and $\tilde{a}_2 = ([a_2, b_2], [c_2, d_2])$ be any two interval-valued intuitionistic fuzzy values, $S(\tilde{a}_1)$ and $S(\tilde{a}_2)$ be the score functions of $\tilde{a}_1$ and $\tilde{a}_2$, respectively, and let $H(\tilde{a}_2)$ and $H(\tilde{a}_2)$ be the accuracy functions of $\tilde{a}$ and $\tilde{b}$, respectively, then

If $S(\tilde{a}_1) < S(\tilde{a}_2)$, then $\tilde{a}_1$ is smaller than $\tilde{a}_2$, denoted by $\tilde{a}_1 < \tilde{a}_2$.
If $S(\tilde{a}_1) = S(\tilde{a}_2)$, then

(1) If $H(\tilde{a}_1) < H(\tilde{a}_2)$, then $\tilde{a}_1$ is smaller than $\tilde{a}_2$, denoted by $\tilde{a}_1 < \tilde{a}_2$;
(2) If $H(\tilde{a}_1) = H(\tilde{a}_2)$, then $\tilde{a}_1$ and $\tilde{a}_2$ represent the same information, denoted by $\tilde{a}_1 = \tilde{a}_2$.

Motivated by the operations in Atanassov (1999, 1994), we define two operational laws of interval-valued intuitionistic fuzzy values.

Definition 8. Let $\tilde{a}_1 = ([a_1, b_1], [c_1, d_1])$ and $\tilde{a}_2 = ([a_2, b_2], [c_2, d_2])$ be two interval-valued intuitionistic fuzzy values, then

(1) $\tilde{a}_1 \cdot \tilde{a}_2 = ([a_1a_2, b_1b_2], [c_1 + c_2 - c_1c_2, d_1 + d_2 - d_1d_2])$;
(2) $\tilde{a}_1^{\lambda} = ([a_1^{\lambda}, b_1^{\lambda}], [1 - (1 - c_1)^{\lambda}, 1 - (1 - d_1)^{\lambda}]),\ \lambda > 0$.

For two operational laws of Definition 8, it is easy to obtain the following these properties.

Proposition 1. *Let $\tilde{a}_1 = ([a_1, b_1], [c_1, d_1])$ and $\tilde{a}_2 = ([a_2, b_2], [c_2, d_2])$ be two interval-valued intuitionistic fuzzy values, and let $\tilde{c} = \tilde{a}_1 \cdot \tilde{a}_2$ and $\tilde{d} = \tilde{a}_1^{\lambda}$, then both $\tilde{c}$ and $\tilde{d}$ are also interval-valued intuitionistic fuzzy values.*

Proposition 2. *Let $\tilde{a}_1 = ([a_1, b_1], [c_1, d_1])$ and $\tilde{a}_2 = ([a_2, b_2], [c_2, d_2])$ be two interval-valued intuitionistic fuzzy values, $\forall \lambda_1, \lambda_2 > 0$. Then we have:*

(1) $\tilde{a}_1 \cdot \tilde{a}_2 = \tilde{a}_2 \cdot \tilde{a}_1$;
(2) $(\tilde{a}_1 \cdot \tilde{a}_2)^{\lambda} = \tilde{a}_1^{\lambda} \cdot \tilde{a}_2^{\lambda}$;
(3) $\tilde{a}_1^{\lambda_1 + \lambda_2} = \tilde{a}_1^{\lambda_1} \cdot \tilde{a}_1^{\lambda_2}$.

4. Generalized interval-valued intuitionistic fuzzy geometric aggregation operator

In the following, based on fuzzy measure, we first give the definition of generalized interval-valued intuitionistic geometric aggregation operator.

Definition 9. Let $\tilde{a}_i = ([a_i, b_i], [c_i, d_i])$ $(i = 1, 2, \ldots, n)$ be a collection of interval-valued intuitionistic fuzzy values on X, and μ be a fuzzy measure on X. Based on fuzzy measure, a generalized interval-valued intuitionistic fuzzy geometric aggregation (GIIFGA) operator of dimension n is a mapping GIIFGA: $\Omega^n \to \Omega$ such that

$$\mathrm{GIIFGA}_{\mu}(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = (\tilde{a}_{(1)})^{\mu(A_{(1)})-\mu(A_{(2)})} \cdot (\tilde{a}_{(2)})^{\mu(A_{(2)})-\mu(A_{(3)})} \cdots (\tilde{a}_{(n)})^{\mu(A_{(n)})-\mu(A_{(n+1)})}, \tag{7}$$

where $_{(\cdot)}$ indicates a permutation on X such that $\tilde{a}_{(1)} \leqslant \tilde{a}_{(2)} \leqslant \cdots \leqslant \tilde{a}_{(n)}$. And $A_{(i)} = ((i), \ldots, (n))$, $A_{(n+1)} = \phi$.

Theorem 1. *Let $\tilde{a}_i = ([a_i, b_i], [c_i, d_i])$ $(i = 1, 2, \ldots, n)$ be a collection of interval-valued intuitionistic fuzzy values on X, and μ be a fuzzy measure on X. then their aggregated value by using the* GIIFGA$_{\mu}$ *operator is also an interval-valued intuitionistic fuzzy value, and*

$$\mathrm{GIIFGA}_{\mu}(\tilde{a}_1, \ldots, \tilde{a}_n) = \left(\left[\prod_{i=1}^{n}(a_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})},\ \prod_{i=1}^{n}(b_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right], \left[1 - \prod_{i=1}^{n}(1 - c_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, 1 - \prod_{i=1}^{n}(1 - d_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right]\right), \tag{8}$$

where $_{(\cdot)}$ indicates a permutation on X such that $\tilde{a}_{(1)} \leqslant \tilde{a}_{(2)} \leqslant \cdots \leqslant \tilde{a}_{(n)}$. And $A_{(i)} = ((i), \ldots, (n))$, $A_{(n+1)} = \phi$.

Proof. The first result follows quickly from Definition 9 and Proposition 1. Below we prove Eq. (8) by using mathematical induction on n.

For $n = 2$, according to the operational laws of Definition 8, we have

$$(\tilde{a}_{(1)})^{\mu(A_{(1)})-\mu(A_{(2)})} = \left(\left[(a_{(1)})^{\mu(A_{(1)})-\mu(A_{(2)})}, (b_{(1)})^{\mu(A_{(1)})-\mu(A_{(2)})}\right], \left[1 - (1 - c_{(1)})^{\mu(A_{(1)})-\mu(A_{(2)})}, 1 - (1 - d_{(1)})^{\mu(A_{(1)})-\mu(A_{(2)})}\right]\right),$$

$$(\tilde{a}_{(2)})^{\mu(A_{(2)})-\mu(A_{(3)})} = \left(\left[(a_{(2)})^{\mu(A_{(1)})-\mu(A_{(2)})}, (b_{(2)})^{\mu(A_{(1)})-\mu(A_{(2)})}\right], \left[1 - (1 - c_{(2)})^{\mu(A_{(1)})-\mu(A_{(2)})}, 1 - (1 - d_{(2)})^{\mu(A_{(1)})-\mu(A_{(2)})}\right]\right).$$

Since

$$\tilde{a}_1 \cdot \tilde{a}_2 = ([a_1a_2, b_1b_2], [c_1 + c_2 - c_1c_2, d_1 + d_2 - d_1d_2]) = ([a_1a_2, b_1b_2],\ [1 - (1 - c_1)(1 - c_2), 1 - (1 - d_1)(1 - d_2)]).$$

Then

$$\begin{aligned}\mathrm{GIIFGA}_{\mu}(\tilde{a}_1, \tilde{a}_2) &= (\tilde{a}_{(1)})^{\mu(A_{(1)})-\mu(A_{(2)})} \cdot (\tilde{a}_{(2)})^{\mu(A_{(2)})-\mu(A_{(3)})} \\ &= ([(a_{(1)})^{\mu(A_{(1)})-\mu(A_{(2)})}(a_{(2)})^{\mu(A_{(2)})-\mu(A_{(3)})}, \\ &\quad (b_{(1)})^{\mu(A_{(1)})-\mu(A_{(2)})}(b_{(2)})^{\mu(A_{(2)})-\mu(A_{(3)})}], \\ &\quad [1 - (1 - c_{(1)})^{\mu(A_{(1)})-\mu(A_{(2)})}(1 - c_{(2)})^{\mu(A_{(2)})-\mu(A_{(3)})}, \\ &\quad 1 - (1 - d_{(1)})^{\mu(A_{(1)})-\mu(A_{(2)})}(1 - d_{(2)})^{\mu(A_{(2)})-\mu(A_{(3)})}]).\end{aligned}$$

That is, for $n = 2$, the Eq. (8) holds.

Suppose that if for $n = k$, Eq. (8) holds, i.e.,

$$\mathrm{GIIFGA}_{\mu}(\tilde{a}_1, \ldots, \tilde{a}_k) = \left(\left[\prod_{i=1}^{k}(a_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})},\ \prod_{i=1}^{k}(b_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right], \left[1 - \prod_{i=1}^{k}(1 - c_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, 1 - \prod_{i=1}^{k}(1 - d_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right]\right).$$

Then, for $n = k + 1$, according to Definition 9, we have

$$\begin{aligned}\mathrm{GIIFGA}_\mu(\tilde a_1,\ldots,\tilde a_{k+1}) &= \Bigg(\Bigg[(a_{(k+1)})^{\mu(A_{(k+1)})-\mu(A_{(k+2)})}\prod_{i=1}^{k}(a_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})},\\ &\quad (b_{(k+1)})^{\mu(A_{(k+1)})-\mu(A_{(k+2)})}\prod_{i=1}^{k}(b_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\Bigg],\\ &\quad \Bigg[1-(1-c_{(k+1)})^{\mu(A_{(k+1)})-\mu(A_{(k+2)})}\prod_{i=1}^{k}(1-c_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})},\\ &\quad 1-(1-d_{(k+1)})^{\mu(A_{(k+1)})-\mu(A_{(k+2)})}\prod_{i=1}^{k}(1-d_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\Bigg]\Bigg)\\ &= \Bigg(\Bigg[\prod_{i=1}^{k+1}(a_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})},\prod_{i=1}^{k+1}(b_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\Bigg],\\ &\quad \Bigg[1-\prod_{i=1}^{k+1}(1-c_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})},\\ &\quad 1-\prod_{i=1}^{k+1}(1-d_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\Bigg]\Bigg).\end{aligned}$$

That is, for $n = k + 1$, Eq. (8) still holds.

Therefore, for all n, the Eq. (8) always holds, which completes the proof of Theorem 1.

Remark 1. Let $\tilde a_i = ([a_i,b_i],[c_i,d_i])$ and $\tilde b_i = ([a'_i,b'_i],\ [c'_i,d'_i])$ $(i = 1, 2,\ldots,n)$ be two collections of interval-valued intuitionistic fuzzy values on X. Since$a_i,b_i,c_i,d_ia'_i,b'_i,c'_i,d'_i \in [0,1]$ for any i, if we assume that $T_P(c_i,c'_i) = c_ic'_i$, $T_P(d_i,d'_i) = d_id'_i, S_P(a_i,a'_i) = a_i + a'_i - a_ia'_i, S_P(b_i, b'_i) = b_i + b'_i - b_ib'_i$, then $T_P(c_i,c'_i)$ and $T_P(d_i,d'_i)$ is one of the basic t-norms, called the product, which is satisfying the following properties (Klement, Mesiar, & Pap, 2000): $T_P(x,1) = x$ (boundary); $T_P(x,y) \leqslant T_P(x,z)$ whenever $y \leqslant z$ (monotonicity); $T_P(x,y) = T_P(y,x)$ (commutativity); $T_P(x,T_P(y,z)) = T_P(T_P(x,y))$ (associativity), where $x,y,z \in [0,1]$. $S_P(a_i,a'_i)$ and $S_P(b_i,b'_i)$ is one of the basic t-conorms, called the probabilistic sum (Klement et al., 2000), andS_P is also called the dual t-conorm of T_P, which is satisfying the boundary, i.e. $S_P(x,0) = x$, monotonicity, commutativity, and associativity (Klement et al., 2000). The associativity of t-norms and t-conorms allows us to extend the product T_P and probabilistic sum S_P in unique way to an n-ary operation in the usual way by induction, defining for each n-tuple $(x_1,x_2,\ldots,x_n) \in [0,1]^n$ and $(y_1,y_2,\ldots,y_n) \in [0,1]^n$, respectively:

$$T_P(x_1,x_2,\ldots,x_n) = \mathop{T_P}_{i=1}^{n} x_i = T_P\left(\mathop{T_P}_{i=1}^{n-1} x_i, x_n\right) = \prod_{i=1}^{n} x_i,$$

$$S_P(y_1,y_2,\ldots,y_n) = \mathop{S_P}_{i=1}^{n} y_i = S_P\left(\mathop{S_P}_{i=1}^{n-1} y_i, y_n\right) = 1-\prod_{i=1}^{n}(1-y_i).$$

Assume that $y_i = 1-(1-c_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}$, $y'_i = 1-(1-d_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}$, $x_i = (a_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, x'_i = (b_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}$, then from Theorem 1 we have

$$\begin{aligned}\mathrm{GIIFGA}_\mu(\tilde a_1,\ldots,\tilde a_n) = (&[T_P(x_1,\ldots,x_n),\ T_P(x'_1,\ldots,x'_n)],\\ &[S_P(y_1,\ldots,y_n),\ S_P(y'_1,\ldots,y'_n)]).\end{aligned}$$

It is shown that the generalized interval-valued intuitionistic fuzzy geometric aggregation operator can be represented by one of the basic t-norms T_P and t-conorms S_P.

Proposition 3. *Let $\tilde a_i = ([a_i,b_i],[c_i,d_i])$ $(i = 1,2,\ldots,n)$ be a collection of interval-valued intuitionistic fuzzy values on X, and μ be a fuzzy measure on X. If all $\tilde a_i$ $(i = 1,2,\ldots,n)$ are equal, that is, for all $i, \tilde a_i = \tilde a = ([a,b],[c,d])$, then*

$$\mathrm{GIIFGA}_\mu(\tilde a_1,\ldots,\tilde a_n) = \tilde a.$$

Proof. According to Theorem 1, if for all i $(i = 1,2,\ldots,n)$, $\tilde a_i = \tilde a$, then

$$\begin{aligned}\mathrm{GIIFGA}_\mu(\tilde a_1,\ldots,\tilde a_n) &= \Bigg(\Bigg[a^{\sum_{i=1}^{n}\mu(A_{(i)})-\mu(A_{(i+1)})}, b^{\sum_{i=1}^{n}\mu(A_{(i)})-\mu(A_{(i+1)})}\Bigg],\\ &\quad\Bigg[1-(1-c)^{\sum_{i=1}^{n}\mu(A_{(i)})-\mu(A_{(i+1)})},\\ &\quad 1-(1-d)^{\sum_{i=1}^{n}\mu(A_{(i)})-\mu(A_{(i+1)})}\Bigg]\Bigg)\end{aligned}$$

Since

$$\sum_{i=1}^{n}(\mu(A_{(i)})-\mu(A_{(i+1)})) = 1.$$

So

$$\mathrm{GIIFGA}_\mu(\tilde a_1,\ldots,\tilde a_n) = ([a,b],[c,d]) = \tilde a. \qquad \square$$

Proposition 4. *Let $\tilde a_i = ([a_i,b_i],[c_i,d_i])$ and $\tilde b_i = ([a'_i,b'_i],[c'_i,d'_i])$ $(i = 1,2,\ldots,n)$ be two collections of interval-valued intuitionistic fuzzy values on X, and μ be a fuzzy measure on X. $_{(\cdot)}$ indicates a permutation on X such that $\tilde a_{(1)} \leqslant \cdots \leqslant \tilde a_{(n)}$ and $\tilde b_{(1)} \leqslant \cdots \leqslant \tilde b_{(n)}$. If $b_{(i)} \leqslant b'_{(i)}$, $a_{(i)} \leqslant a'_{(i)}$ and $d_{(i)} \geqslant d'_{(i)}$, $c_{(i)} \geqslant c'_{(i)}$ for all i, that is, $\tilde a_{(i)} \leqslant \tilde b_{(i)}$, then*

$$\mathrm{GIIFGA}_\mu(\tilde a_1,\ldots,\tilde a_n) \leqslant \mathrm{GIIFGA}_\mu(\tilde b_1,\ldots,\tilde b_n).$$

Proof. Since $A_{(i+1)} \subseteq A_{(i)}$, then $\mu(A_{(i)}) - \mu(A_{(i+1)}) \geqslant 0$. For all i, $b_{(i)} \leqslant b'_{(i)}$, $a_{(i)} \leqslant a'_{(i)}$ and $d_{(i)} \geqslant d'_{(i)}$, $c_{(i)} \geqslant c'_{(i)}$, we have

$$\prod_{i=1}^{n}(a_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})} \leqslant \prod_{i=1}^{n}\left(a'_{(i)}\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})},$$

$$\prod_{i=1}^{n}(b_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})} \leqslant \prod_{i=1}^{n}\left(b'_{(i)}\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})},$$

$$1-\prod_{i=1}^{n}(1-c_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})} \geqslant 1-\prod_{i=1}^{n}\left(1-c'_{(i)}\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})},$$

$$1-\prod_{i=1}^{n}(1-d_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})} \geqslant 1-\prod_{i=1}^{n}\left(1-d'_{(i)}\right)^{\mu(A_{(i)})-\mu(A_{(i+1)})}. \qquad \square$$

According to Theorem 1 and Eq. (6), we have

$$\mathrm{GIIFGA}_\mu(\tilde a_1,\ldots,\tilde a_n) \leqslant \mathrm{GIIFGA}_\mu(\tilde b_1,\ldots,\tilde b_n).$$

Proposition 5. *Let $\tilde a_i = ([a_i,b_i],[c_i,d_i])$ $(i = 1,2,\ldots,n)$ be a collection of interval-valued intuitionistic fuzzy values on X, and μ be a fuzzy measure on X. If*

$$\tilde a^- = \left(\left[\min_i a_i, \min_i b_i\right], \left[\max_i c_i, \max_i d_i\right]\right),$$

$$\tilde a^+ = \left(\left[\max_i a_i, \max_i b_i\right], \left[\min_i c_i, \min_i d_i\right]\right),$$

then

$$\tilde a^- \leqslant \mathrm{GIIFGA}_\mu(\tilde a_1,\ldots,\tilde a_n) \leqslant \tilde a^+.$$

Proof. For any $\tilde a_i = ([a_i,b_i],[c_i,d_i])$ $(i = 1,\ldots,n)$, it is obvious that $\tilde a^-$ and $\tilde a^+$ are interval-valued intuitionistic fuzzy values. Since $A_{(i+1)} \subseteq A_{(i)}$, then $\mu(A_{(i)}) - \mu(A_{(i+1)}) \geqslant 0$.

Let $_{(\cdot)}$ indicate a permutation on X such that$\tilde a_{(1)} \leqslant \cdots \leqslant \tilde a_{(n)}$, we have

$$\begin{aligned}&\min_i a_i \leqslant a_{(i)} \leqslant \max_i a_i, \min_i b_i \leqslant b_{(i)} \leqslant \max_i b_i, \min_i c_i \leqslant c_{(i)}\\ &\quad\leqslant \max_i c_i, \min_i d_i \leqslant d_{(i)} \leqslant \max_i d_i.\end{aligned}$$

Thus,

$$\prod_{i=1}^{n}(\min_i a_i)^{\mu(A_{(i)})-\mu(A_{(i+1)})} \leqslant \prod_{i=1}^{n}(a_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})} \leqslant \prod_{i=1}^{n}(\max_i a_i)^{\mu(A_{(i)})-\mu(A_{(i+1)})},$$

$$\prod_{i=1}^{n}(\min_i b_i)^{\mu(A_{(i)})-\mu(A_{(i+1)})} \leqslant \prod_{i=1}^{n}(b_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})} \leqslant \prod_{i=1}^{n}(\max_i b_i)^{\mu(A_{(i)})-\mu(A_{(i+1)})},$$

and

$$1-\prod_{i=1}^{n}(1-\min_i c_i)^{\mu(A_{(i)})-\mu(A_{(i+1)})} \leqslant 1-\prod_{i=1}^{n}(1-c_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})} \leqslant 1-\prod_{i=1}^{n}(1-\max_i c_i)^{\mu(A_{(i)})-\mu(A_{(i+1)})},$$

$$1-\prod_{i=1}^{n}(1-\min_i d_i)^{\mu(A_{(i)})-\mu(A_{(i+1)})} \leqslant 1-\prod_{i=1}^{n}(1-d_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})} \leqslant 1-\prod_{i=1}^{n}(1-\max_i d_i)^{\mu(A_{(i)})-\mu(A_{(i+1)})}.$$

i.e.,

$$(\min_i a_i)^{\sum_{i=1}^{n}\mu(A_{(i)})-\mu(A_{(i+1)})} \leqslant \prod_{i=1}^{n}(a_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})} \leqslant (\max_i a_i)^{\sum_{i=1}^{n}\mu(A_{(i)})-\mu(A_{(i+1)})},$$

$$(\min_i b_i)^{\sum_{i=1}^{n}\mu(A_{(i)})-\mu(A_{(i+1)})} \leqslant \prod_{i=1}^{n}(b_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})} \leqslant (\max_i b_i)^{\sum_{i=1}^{n}\mu(A_{(i)})-\mu(A_{(i+1)})},$$

and

$$1-(1-\min_i c_i)^{\sum_{i=1}^{n}\mu(A_{(i)})-\mu(A_{(i+1)})} \leqslant 1-\prod_{i=1}^{n}(1-c_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})} \leqslant 1-(1-\max_i c_i)^{\sum_{i=1}^{n}\mu(A_{(i)})-\mu(A_{(i+1)})},$$

$$1-(1-\min_i d_i)^{\sum_{i=1}^{n}\mu(A_{(i)})-\mu(A_{(i+1)})} \leqslant 1-\prod_{i=1}^{n}(1-d_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})} \leqslant 1-(1-\max_i d_i)^{\sum_{i=1}^{n}\mu(A_{(i)})-\mu(A_{(i+1)})}.$$

Since

$$\sum_{i=1}^{n}(\mu(A_{(i)})-\mu(A_{(i+1)}))=1.$$

So we have

$$\min_i a_i \leqslant \prod_{i=1}^{n}(a_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})} \leqslant \max_i a_i,$$

$$\min_i b_i \leqslant \prod_{i=1}^{n}(b_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})} \leqslant \max_i b_i,$$

$$\min_i c_i \leqslant 1-\prod_{i=1}^{n}(1-c_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})} \leqslant \max_i c_i,$$

$$\min_i d_i \leqslant 1-\prod_{i=1}^{n}(1-d_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})} \leqslant \max_i d_i.$$

According to Theorem 1 and Eq. (6), we have

$$\left([\min_i a_i, \min_i b_i],[\max_i c_i, \max_i d_i]\right) \leqslant \text{GIIFGA}_\mu(\tilde{a}_1,\ldots,\tilde{a}_n) \leqslant ([\max_i a_i, \max_i b_i],[\min_i c_i, \min_i d_i]),$$

That is, $\tilde{a}^- \leqslant \text{GIIFGA}_\mu(\tilde{a}_1,\ldots,\tilde{a}_n) \leqslant \tilde{a}^+$. □

Proposition 6. *Let $\tilde{a}_i = ([a_i,b_i],[c_i,d_i])$ $(i=1,2,\ldots,n)$ be a collection of interval-valued intuitionistic fuzzy values on X, and μ be a fuzzy measure on X. If $\tilde{s}=([a,b],[c,d])$ is an interval-valued intuitionistic fuzzy value on X, then*

$$\text{GIIFGA}_\mu(\tilde{a}_1\cdot\tilde{s},\ldots,\tilde{a}_n\cdot\tilde{s}) = \text{GIIFGA}_\mu(\tilde{a}_1,\ldots,\tilde{a}_n)\cdot\tilde{s}.$$

Proof. Since for any i $(i = 1,2,\ldots,n)$

$$\tilde{a}_i\cdot\tilde{s} = ([a_ia, b_ib],[c_i+c-c_ic, d_i+d-d_id]) = ([a_ia,b_ib],[1-(1-c_i)(1-c),1-(1-d_i)(1-d)]).$$

According to Theorem 1, we have

$$\begin{aligned}\text{GIIFGA}_\mu(\tilde{a}_1\cdot\tilde{s},\ldots,\tilde{a}_n\cdot\tilde{s}) &= \Bigg(\Bigg[\prod_{i=1}^{n}(a_{(i)}a)^{\mu(A_{(i)})-\mu(A_{(i+1)})}, \prod_{i=1}^{n}(b_{(i)}b)^{\mu(A_{(i)})-\mu(A_{(i+1)})}\Bigg], \\ &\quad \Bigg[1-\prod_{i=1}^{n}((1-c_{(i)})(1-c))^{\mu(A_{(i)})-\mu(A_{(i+1)})}, 1-\prod_{i=1}^{n}((1-d_{(i)})(1-d))^{\mu(A_{(i)})-\mu(A_{(i+1)})}\Bigg]\Bigg) \\ &= \Bigg(\Bigg[a\prod_{i=1}^{n}(a_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, b\prod_{i=1}^{n}(d_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\Bigg], \\ &\quad \Bigg[1-(1-c)\prod_{i=1}^{n}(1-c_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, 1-(1-d)\prod_{i=1}^{n}(1-d_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\Bigg]\Bigg).\end{aligned}$$

According to Definition 8, we have

$$\begin{aligned}\text{GIIFGA}_\mu(\tilde{a}_1,\ldots,\tilde{a}_n)\cdot\tilde{s} &= \Bigg(\Bigg[a\prod_{i=1}^{n}(a_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, b\prod_{i=1}^{n}(b_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\Bigg], \\ &\quad \Bigg[1-(1-c)\prod_{i=1}^{n}(1-c_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}, 1-(1-d)\prod_{i=1}^{n}(1-d_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\Bigg]\Bigg).\end{aligned}$$

Thus,

$$\text{GIIFGA}_\mu(\tilde{a}_1\cdot\tilde{s},\ldots,\tilde{a}_n\cdot\tilde{s}) = \text{GIIFGA}_\mu(\tilde{a}_1,\ldots,\tilde{a}_n)\cdot\tilde{s}.$$ □

Proposition 7. *Let $\tilde{a}_i = ([a_i,b_i],[c_i,d_i])$ $(i=1,2,\ldots,n)$ be a collection of interval-valued intuitionistic fuzzy values on X, and μ be a fuzzy measure on X. If $r > 0$, then*

$$\text{GIIFGA}_\mu((\tilde{a}_1)^r,\ldots,(\tilde{a}_n)^r) = (\text{GIIFGA}_\mu(\tilde{a}_1,\ldots,\tilde{a}_n))^r.$$

Proof. According to Definition 8, for any i $(i = 1,2,\ldots,n)$ and $r > 0$ we have.

According to Theorem 1, we have

$$\begin{aligned}\mathrm{GIIFGA}_\mu((\tilde a_1)^r,\ldots,(\tilde a_n)^r) &= \Bigg(\Bigg[\prod_{i=1}^n((a_{(i)})^r)^{(\mu(A_{(i)})-\mu(A_{(i+1)}))},\\ &\prod_{i=1}^n((b_{(i)})^r)^{(\mu(A_{(i)})-\mu(A_{(i+1)}))},\Bigg]\\ &\Bigg[1-\prod_{i=1}^n((1-c_{(i)})^r)^{\mu(A_{(i)})-\mu(A_{(i+1)})},\\ &1-\prod_{i=1}^n((1-d_{(i)})^r)^{\mu(A_{(i)})-\mu(A_{(i+1)})}\Bigg]\Bigg)\\ &= \Bigg(\Bigg[\prod_{i=1}^n(a_{(i)})^{r(\mu(A_{(i)})-\mu(A_{(i+1)}))},\\ &\prod_{i=1}^n(b_{(i)})^{r(\mu(A_{(i)})-\mu(A_{(i+1)}))}\Bigg],\\ &\Bigg[1-\prod_{i=1}^n(1-c_{(i)})^{r(\mu(A_{(i)})-\mu(A_{(i+1)}))},\\ &1-\prod_{i=1}^n(1-d_{(i)})^{r(\mu(A_{(i)})-\mu(A_{(i+1)}))}\Bigg]\Bigg).\end{aligned}$$

Since

$$\begin{aligned}(\mathrm{GIIFGA}_\mu(\tilde a_1,\ldots,\tilde a_n))^r &= \Bigg(\Bigg[(\prod_{i=1}^n(a_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})})^r,\\ &\left(\prod_{i=1}^n(b_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right)^r\Bigg],\\ &\Bigg[1-\left(\prod_{i=1}^n(1-c_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right)^r,\\ &1-\left(\prod_{i=1}^n(1-d_{(i)})^{\mu(A_{(i)})-\mu(A_{(i+1)})}\right)^r\Bigg]\Bigg)\\ &= \Bigg(\Bigg[\prod_{i=1}^n(a_{(i)})^{r(\mu(A_{(i)})-\mu(A_{(i+1)}))},\prod_{i=1}^n(b_{(i)})^{r(\mu(A_{(i)})-\mu(A_{(i+1)}))}\Bigg],\\ &\Bigg[1-\prod_{i=1}^n(1-c_{(i)})^{r(\mu(A_{(i)})-\mu(A_{(i+1)}))},\\ &1-\prod_{i=1}^n(1-d_{(i)})^{r(\mu(A_{(i)})-\mu(A_{(i+1)}))}\Bigg]\Bigg).\end{aligned}$$

Thus,

$$\mathrm{GIIFGA}_\mu((\tilde a_1)^r,\ldots,(\tilde a_n)^r) = (\mathrm{GIIFGA}_\mu(\tilde a_1,\ldots,\tilde a_n))^r. \quad \square$$

According to Propositions 6 and 7, we can obtain the following corollary.

Corollary 1. *Let* $\tilde a_i = ([a_i,b_i],[c_i,d_i])$ $(i=1,2,\ldots,n)$ *be a collection of interval-valued intuitionistic fuzzy values on X, and* μ *be a fuzzy measure on X. If* $r>0$ *and* $\tilde s = ([a,b],[c,d])$ *is an interval-valued intuitionistic fuzzy value on X, then*
$\mathrm{GIIFGA}_\mu((\tilde a_1)^r\cdot\tilde s,\ldots,(\tilde a_n)^r\cdot\tilde s) = (\mathrm{GIIFGA}_\mu(\tilde a_1,\tilde a_2,\ldots,\tilde a_n))^r\cdot\tilde s.$

According to Theorem 1, it is easily obtained the following conclusion.

Proposition 8. *Let* $\tilde a_i = ([a_i,b_i],[c_i,d_i])$ $(i=1,2,\ldots,n)$ *be a collection of interval-valued intuitionistic fuzzy values on X, and* μ *be a fuzzy measure on X.*

(1) *If* $\mu(A)=1$ *for any* $A\in P(X)$, *then* $\mathrm{GIIFGA}_\mu(\tilde a_1,\ldots,\tilde a_n) = max(\tilde a_1,\ldots,\tilde a_n) = \tilde a_{(n)}$
(2) *If* $\mu(A)=0$ *for any* $A\in P(X)$ *and* $A\neq X$, *then* $\mathrm{GIIFGA}_\mu(\tilde a_1,\ldots,\tilde a_n) = min(\tilde a_1,\ldots,\tilde a_n) = \tilde a_{(1)}$.
(3) *For any* $A,\ B\in P(X)$ *such that* $|A|=|B|$, *if* $\mu(A)=\mu(B)$ *and* $\mu\{(i),\ldots,(n)\} = \frac{n-i+1}{n}$, $1\leqslant i\leqslant n$, *then*

$$\mathrm{GIIFGA}_\mu(\tilde a_1,\ldots,\tilde a_n) = \left(\left[\prod_{i=1}^n(a_i)^{\frac{1}{n}},\prod_{i=1}^n(b_i)^{\frac{1}{n}}\right],\ \left[1-\prod_{i=1}^n(1-c_i)^{\frac{1}{n}},\ 1-\prod_{i=1}^n(1-d_i)^{\frac{1}{n}}\right]\right).$$

According to Definitions 7 and 8, Xu (2007) proposed interval-valued intuitionistic fuzzy ordered weighted geometric averaging (IIFOWGA) operator, which are defined as follows.

Definition 10. Let $\tilde a_i = ([a_i,b_i],[c_i,d_i])$ $(i=1,2,\ldots,n)$ be a collection of interval-valued intuitionistic fuzzy values on X. An interval-valued intuitionistic fuzzy ordered weighted geometric averaging (IIFOWGA) operator of dimension n is a mapping IIFOWGA: $\Omega^n\to\Omega$, that has associated with it an exponential weighting vector $w=(w_1,w_2,\ldots,w_n)^T$, with $w_i\in[0,1]$ and $\sum_{i=1}^n w_i = 1$, such that

$$\mathrm{IIFOWGA}_w(\tilde a_1,\tilde a_2,\ldots,\tilde a_n) = (\tilde a_{(1)})^{w_1}\cdot(\tilde a_{(2)})^{w_2}\cdots(\tilde a_{(n)})^{w_n},$$

where $_{(\cdot)}$ indicates is a permutation on X such that $\tilde a_{(1)}\leqslant\tilde a_{(2)}\leqslant\cdots\leqslant\tilde a_{(n)}$. Furthermore

$$\mathrm{IIFOWGA}_w(\tilde a_1,\tilde a_2,\ldots,\tilde a_n) = \left(\left[\prod_{i=1}^n(a_{(i)})^{w_i},\prod_{i=1}^n(b_{(i)})^{w_i}\right],\left[1-\prod_{i=1}^n(1-c_{(i)})^{w_i},1-\prod_{i=1}^n(1-d_{(i)})^{w_i}\right]\right).$$

In the following, we will find some relations between GIIFGA and IIFOWGA operators.

Suppose that IIFOWGA operator has associated with it an exponential weighting vector $w=(w_1,w_2,\ldots,w_n)^T$, according to Theorem 1 and Definition 10, it is easily seen that IIFOWGA operator will be equivalent to a GIIFGA operator, where fuzzy measure μ associated to the GIIFGA is given by

$$\mu(S) = \sum_{i=n-s+1} w_i (S\subseteq X, S\neq\phi).$$

Conversely, the GIIFGA operator will be equivalent to the IIFOWGA operator that has associated with it an exponential weighting vector $w=(w_1,w_2,\ldots,w_n)^T$, $w_{n-s} = \mu(S\cup i)-\mu(S)$, $i\in X$, $S\subseteq X\setminus i$.

According to above analysis, it is easily obtained the following conclusion.

Theorem 2. *Let* $\tilde a_i = ([a_i,b_i],[c_i,d_i])$ $(i=1,2,\ldots,n)$ *be a collection of interval-valued intuitionistic fuzzy values on X, and* μ *be a fuzzy measure on X. The following assertions are equivalent:*

(1) *For any* $A,B\in P(X)$ *such that* $|A|=|B|$, *we have* $\mu(A)=\mu(B)$.
(2) *There exists an exponential weighting vector* $w=(w_1,w_2,\ldots,w_n)$ *such that*

$$\mathrm{GIIFGA}_\mu(\tilde a_1,\tilde a_2,\ldots,\tilde a_n) = \mathrm{IIFOWGA}_w(\tilde a_1,\tilde a_2,\ldots,\tilde a_n).$$

(3) *GIFOGA is a symmetric function.*

Proof. The proof is similar to that of Proposition 4.1 in Marichal (2002). Here we do not duplicate it.

Suppose that IIFWGA operator has an exponential weighting vector $w=(w_1,w_2,\ldots,w_n)^T$, according to Theorem 1, it is easily seen that if μ is an additive fuzzy measure, GIIFGA operator will be equivalent to a IIFWGA operator, where $w_i=\mu(i)$. Reciprocally, one can readily see that IIFWGA is a GIIFGA operator which has an additive fuzzy measure μ:

$$\mu(A) = \sum_{i \in A} w_i, \quad (A \subseteq X).$$

From Theorem 2 and above analysis, it is easily known that the GIIFOGA operator generalizes both IIFOWGA and IIFWGA operators. The IIFOWGA and IIFWGA operators are two special cases of GIIFOGA operator.

5. Choquet integral-based TOPSIS for multi-criteria interval-valued intuitionistic fuzzy group decision making

As generalization of the linear Lebesgue integral (e.g. weighted average method), Choquet integral is defined as follows.

Definition 11. Grabisch et al., 2000 Let $X = \{x_1, x_2, \ldots, x_n\}$ be a universe of discourse, f be a positive real-valued function on X, and μ be a fuzzy measure on X. The discrete Choquet integral of f with respective to μ is defined by

$$C_\mu(f) = \sum_{i=1}^{n} f(x_{(i)})[\mu(A_{(i)}) - \mu(A_{(i+1)})],$$

where $_{(\cdot)}$ indicates a permutation on X such that $f(x_{(1)}) \leqslant f(x_{(2)}) \leqslant \cdots \leqslant f(x_{(n)})$. Also $A_{(i)} = \{x_{(i)}, \ldots, x_{(n)}\}$, $A_{(n+1)} = \phi$.

The main advantage of the Choquet integral is that it coincides with the Lebesgue integral when the measure is additive. An additive measure may be directly tied to the notions of additive expected utility (Schmeidler, 1989) and mutual preferential independence (Marichal, 1998). The Choquet integral is able to perform aggregation of criteria even when mutual preferential independence is violated.

Inspired by Definition 11, Choquet integral-based Hamming distance between two interval-valued intuitionistic fuzzy values is defined as follows.

Definition 12. Let $\tilde{a} = ([a_i, b_i], [c_i, d_i])$ and $\tilde{b} = ([a'_i, b'_i], [c'_i, d'_i])$ $(i = 1, 2, \ldots, n)$ be two interval-valued intuitionistic fuzzy values on X, and μ be a fuzzy measure on X. $d(\tilde{a}, \tilde{b})$ is defined by Choquet integral-based Hamming distance as

$$d(\tilde{a}, \tilde{b}) = \frac{1}{4} \sum_{i=1}^{n} d_{(i)}(\tilde{a}, \tilde{b})(\mu(A_{(i)}) - \mu(A_{(i+1)})),$$

where $d_i(\tilde{a}, \tilde{b}) = |a_i - a'_i| + |b_i - b'_i| + |c_i - c'_i| + |d_i - d'_i|$, so that $d_{(1)}(\tilde{a}, \tilde{b}) \leqslant d_{(2)}(\tilde{a}, \tilde{b}) \leqslant \cdots \leqslant d_{(n)}(\tilde{a}, \tilde{b})$, $A_{(i)} = \{x_{(i)}, \ldots, x_{(n)}\}$, $A_{(n+1)} = \phi$.

In general, multi-criteria group decision making problem includes uncertain and imprecise data and information. We consider the multi-criteria group decision making problems where all the criteria values are expressed in interval-valued intuitionistic fuzzy values, and interactions phenomena among the decision making criteria or preference of decision makers are taken into account. The following notations are used to depict the considered problems:

$E = (e_1, e_2, \ldots, e_r)$ is the set of the experts involved in the decision process;
$A = (a_1, a_2, \ldots, a_m)$ is the set of the considered alternatives;
$C = (c_1, c_2, \ldots, c_n)$ is the set of the criteria used for evaluating the alternatives.

In group decision making problems, aggregation of expert opinions is very important to appropriately perform evaluation process. In the following, according to Choquet integral-based Hamming distance, Choquet integral-based TOPSIS is proposed for multi-criteria interval-valued intuitionistic fuzzy group decision making where expert opinions are aggregated by the generalized interval-valued intuitionistic fuzzy geometric aggregation operator, which involves the following steps:

Step 1. As for every alternative a_i $(i = 1, 2, \ldots, m)$, each expert e_k $(k = 1, 2, \ldots, r)$ is invited to express their individual evaluation or preference according to each criteria c_j $(j = 1, 2, \ldots, n)$ by an interval-valued intuitionistic fuzzy value $\tilde{a}^k_{ij} = ([a_{\tilde{a}^k_{ij}}, b_{\tilde{a}^k_{ij}}], [c_{\tilde{a}^k_{ij}}, d_{\tilde{a}^k_{ij}}])$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n,\ k = 1, 2, \ldots, r)$, where $[a_{\tilde{a}^k_{ij}}, b_{\tilde{a}^k_{ij}}]$ indicates the uncertain degree that expert e_k considers what the alternative a_i should satisfy the criteria c_j, $[c_{\tilde{a}^k_{ij}}, d_{\tilde{a}^k_{ij}}]$ indicates the uncertain degree that expert e_k considers what the alternative a_i should not satisfy the criteria c_j. Then we can obtain a decision making matrix as follow:

$$R^k = \begin{pmatrix} \tilde{a}^k_{11}, \tilde{a}^k_{12}, \ldots, \tilde{a}^k_{1n} \\ \tilde{a}^k_{21}, \tilde{a}^k_{22}, \ldots, \tilde{a}^k_{2n} \\ \cdots \\ \tilde{a}^k_{m1}, \tilde{a}^k_{m2}, \ldots, \tilde{a}^k_{mn} \end{pmatrix}.$$

Step 2. Confirm the fuzzy density $g_i = g(e_i)$ of each expert. According to Eq. (4), parameter λ_1 of expert can be determined.

Step 3. By Eq. (6) or Definition 7, $\tilde{a}^k_{ij}$ is reordered such that $\tilde{a}^{(k)}_{ij} \leqslant \tilde{a}^{(k+1)}_{ij}$.
Utilize the interval-valued intuitionistic fuzzy Choquet integral operator

$$\tilde{a}_{ij} = \mathrm{GIIFGA}_g(\tilde{a}^1_{ij}, \ldots, \tilde{a}^r_{ij}) = \left(\left[\prod_{k=1}^{r} (a_{\tilde{a}^{(k)}_{ij}})^{g(A_{(k)}) - \mu(A_{(k+1)})}, \prod_{k=1}^{r} (b_{\tilde{a}^{(k)}_{ij}})^{g(A_{(k)}) - \mu(A_{(k+1)})} \right], \left[1 - \prod_{k=1}^{r} (1 - c_{\tilde{a}^{(k)}_{ij}})^{g(A_{(k)}) - g(A_{(k+1)})}, 1 - \prod_{k=1}^{r} (1 - d_{\tilde{a}^{(k)}_{ij}})^{g(A_{(k)}) - g(A_{(k+1)})} \right] \right)$$

to aggregate all the interval-valued intuitionistic fuzzy decision matrices $R^k = (\tilde{a}^k_{ij})_{m \times n}$ $(k = 1, 2, \ldots, r)$ into a complex interval-valued intuitionistic fuzzy decision matrix $R = (\tilde{a}_{ij})_m \times n$, where $\tilde{a}_{ij} = ([a_{\tilde{a}_{ij}}, b_{\tilde{a}_{ij}}], [c_{\tilde{a}_{ij}}, d_{\tilde{a}_{ij}}])$ $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n)$, $A_{(k)} = \{e_{(k)}, \ldots, e_{(r)}\}$, $A_{(r+1)} = \phi$, and $g(A_{(k)})$ can be calculated by Eq. (3).

Step 4. Let J_1 be a collection of benefit criteria (i.e., the larger c_j, the greater preference) and J_2 be a collection of cost criteria (i.e., the smaller c_j, the greater preference). The interval-valued intuitionistic fuzzy positive-ideal solution (IV-IFPIS), denoted as $\tilde{\alpha}^+$, and the interval-valued intuitionistic fuzzy negative-ideal solution (IV-IFNIS), denoted as $\tilde{\alpha}^- = (\tilde{\alpha}^-_1, \tilde{\alpha}^-_2, \ldots, \tilde{\alpha}^-_n)$, are defined as follows:

$$\tilde{\alpha}^+ = \left\{ \left\langle c_j, \left(\left[(\max_i a_{\tilde{a}_{ij}}, \max_i b_{\tilde{a}_{ij}}) | j \in J_1, (\min_i a_{\tilde{a}_{ij}}, \min_i b_{\tilde{a}_{ij}}) \middle| j \in J_2 \right] \right), \left(\left[(\min_i c_{\tilde{a}_{ij}}, \min_i d_{\tilde{a}_{ij}}) | j \in J_1, (\max_i c_{\tilde{a}_{ij}}, \max_i d_{\tilde{a}_{ij}}) | j \in J_2 \right] \right) \right\rangle \middle| i = 1, 2, \ldots, m \right.$$

where $\tilde{\alpha}^+_j = ([a_{\tilde{\alpha}^+_j}, b_{\tilde{\alpha}^+_j}], [c_{\tilde{\alpha}^+_j}, d_{\tilde{\alpha}^+_j}])$ $(j = 1, 2, \ldots, n)$.

$$\tilde{\alpha}^- = \left\{ \left\langle c_j, \left(\left[(\min_i a_{\tilde{a}_{ij}}, \min_i b_{\tilde{a}_{ij}}) \middle| j \in J_1, (\max_i a_{\tilde{a}_{ij}}, \max_i b_{\tilde{a}_{ij}}) \middle| j \in J_2 \right], \left[(\max_i c_{\tilde{a}_{ij}}, \max_i d_{\tilde{a}_{ij}}) \middle| j \in J_1, (\min_i c_{\tilde{a}_{ij}}, \min_i d_{\tilde{a}_{ij}}) \middle| j \in J_2 \right] \right) \right\rangle \middle| i = 1, 2, \ldots, m \right\} = (\tilde{\alpha}^-_1, \tilde{\alpha}^-_2, \ldots, \tilde{\alpha}^-_n),$$

where $\tilde{\alpha}_j^- = ([a_{\tilde{\alpha}_j^-}, b_{\tilde{\alpha}_j^-}], [c_{\tilde{\alpha}_j^-}, d_{\tilde{\alpha}_j^-}])$ $(j = 1, 2, \ldots, n)$.

Moreover, we denote the alternatives a_i $(i = 1, 2, \ldots, m)$ by $x_i = (\tilde{a}_{i1}, \tilde{a}_{i2}, \ldots, \tilde{a}_{in})$.

Step 5. Confirm the fuzzy density $g_i = g(c_i)$ of each criteria. According to Eq. (4), parameter λ_2 of criteria can be determined.

Step 6. According to Choquet integral-based Hamming distance, calculate the distance between the alternative x_i and the IV-IFPIS $\tilde{\alpha}^+$ and the distance between the alternative x_i and the IV-IFNIS $\tilde{\alpha}^-$, respectively:

$$d_i(x_i, \tilde{\alpha}^+) = \frac{1}{4}\sum_{j=1}^{n} d_{i(j)}(\tilde{a}_{ij}, \tilde{\alpha}_j^+)(g(A_{(j)}) - g(A_{(j+1)})), \tag{9}$$

where $d_{ij}(\tilde{a}_{ij}, \tilde{\alpha}_j^+) = \left|a_{\tilde{\alpha}_j^+} - a_{\tilde{a}_{ij}}\right| + \left|b_{\tilde{\alpha}_j^+} - b_{\tilde{a}_{ij}}\right| + \left|c_{\tilde{\alpha}_j^+} - c_{\tilde{a}_{ij}}\right| + \left|d_{\tilde{\alpha}_j^+} - d_{\tilde{a}_{ij}}\right|$, so that $d_{i(1)}(\tilde{a}_{ij}, \tilde{\alpha}_j^+) \leqslant d_{i(2)}(\tilde{a}_{ij}, \tilde{\alpha}_j^+) \leqslant \cdots \leqslant d_{i(n)}(\tilde{a}_{ij}, \tilde{\alpha}_j^+)$, $A_{(j)} = \{c_{(j)}, \ldots, c_{(n)}\}$, $A_{(n+1)} = \phi$. $g(A_{(j)})$ can be calculated by Eq. (3)

$$d_i(x_i, \tilde{\alpha}^-) = \frac{1}{4}\sum_{j=1}^{n} d_{ij}(\tilde{a}_{ij}, \tilde{\alpha}_j^-)(g(A_{(j)}) - g(A_{(j+1)})), \tag{10}$$

where $d_{ij}\left(\tilde{a}_{ij}, \tilde{\alpha}_j^-\right) = \left|a_{\tilde{\alpha}_j^-} - a_{\tilde{a}_{ij}}\right| + \left|b_{\tilde{\alpha}_j^-} - b_{\tilde{a}_{ij}}\right| + \left|c_{\tilde{\alpha}_j^-} - c_{\tilde{a}_{ij}}\right| + \left|d_{\tilde{\alpha}_j^-} - d_{\tilde{a}_{ij}}\right|$, so that $d_{i(1)}(\tilde{a}_{ij}, \tilde{\alpha}_j^-) \leqslant d_{i(2)}(\tilde{a}_{ij}, \tilde{\alpha}_j^-) \leqslant \cdots \leqslant d_{i(n)}(\tilde{a}_{ij}, \tilde{\alpha}_j^-)$, $A_{(j)} = \{c_{(j)}, \ldots, c_{(n)}\}$, $A_{(n+1)} = \phi$. $g(A_{(k)})$ can be calculated by Eq. (3).

Step 7. Calculate the closeness coefficient of each alternative:

$$r(x_i) = \frac{d_i(x_i, \tilde{\alpha}^-)}{d_i(x_i, \tilde{\alpha}^+) + d_i(x_i, \tilde{\alpha}^-)}, \quad i = 1, 2, \ldots, m. \tag{11}$$

Step 8. Rank all the alternatives a_i $(i = 1, 2, \ldots, m)$ according to the closeness coefficients $r(x_i)$, the greater the value $r(x_i)$, the better the alternative a_i.

Step 9. End.

The main difference between the traditional TOPSIS and Choquet integral-based TOPSIS (CITOPSIS) is that the CITOPSIS takes the Choquet Integral-based Hamming distance into account. It is reasonable to employ the Choquet integral in terms of the fuzzy measure to aggregate the performance values instead of the weighted average method, since the Choquet integral does not assume the independence of one element from another.

6. A numerical example

Let us suppose there is an investment company, which wants to invest a sum of money in the best option (adapted from Ref. Schmeidler, 1989). There is a panel with five possible alternatives to invest the money: a_1 is a car company; a_2 is a food company; a_3 is a computer company; a_4 is an arms company; a_5 is a TV company. The investment company must take a decision according to the following four criteria: c_1 is the risk analysis; c_2 is the growth analysis; c_3 is the social–political impact analysis; c_4 is the environmental impact analysis. The five possible alternatives $a_i (i = 1, 2, 3, 4, 5)$ are to be evaluated using the interval-valued intuitionistic fuzzy information by three decision makers e_k $(k = 1, 2, 3)$, as listed in the following matrix

$$R^1 = \begin{pmatrix} ([0.4,0.5],[0.3,0.4]) & ([0.4,0.6],[0.2,0.4]) & ([0.1,0.3],[0.5,0.6]) & ([0.3,0.4],[0.3,0.5]) \\ ([0.6,0.7],[0.2,0.3]) & ([0.6,0.7],[0.2,0.3]) & ([0.4,0.7],[0.1,0.2]) & ([0.5,0.6],[0.1,0.3]) \\ ([0.6,0.7],[0.1,0.2]) & ([0.5,0.6],[0.3,0.4]) & ([0.5,0.6],[0.1,0.3]) & ([0.4,0.5],[0.2,0.4]) \\ ([0.3,0.4],[0.2,0.3]) & ([0.6,0.7],[0.1,0.3]) & ([0.3,0.4],[0.1,0.2]) & ([0.3,0.7],[0.1,0.2]) \\ ([0.7,0.8],[0.1,0.2]) & ([0.3,0.5],[0.1,0.3]) & ([0.5,0.6],[0.2,0.3]) & ([0.3,0.4],[0.5,0.6]) \end{pmatrix},$$

$$R^2 = \begin{pmatrix} ([0.3,0.4],[0.4,0.5]) & ([0.5,0.6],[0.1,0.3]) & ([0.4,0.5],[0.3,0.4]) & ([0.4,0.6],[0.2,0.4]) \\ ([0.3,0.6],[0.3,0.4]) & ([0.4,0.7],[0.1,0.2]) & ([0.5,0.6],[0.2,0.3]) & ([0.6,0.7],[0.2,0.3]) \\ ([0.6,0.8],[0.1,0.2]) & ([0.5,0.6],[0.1,0.2]) & ([0.5,0.7],[0.2,0.3]) & ([0.1,0.3],[0.5,0.6]) \\ ([0.4,0.5],[0.3,0.5]) & ([0.5,0.8],[0.1,0.2]) & ([0.2,0.5],[0.3,0.4]) & ([0.4,0.7],[0.1,0.2]) \\ ([0.6,0.7],[0.2,0.3]) & ([0.6,0.7],[0.1,0.2]) & ([0.5,0.7],[0.2,0.3]) & ([0.6,0.7],[0.1,0.3]) \end{pmatrix},$$

$$R^3 = \begin{pmatrix} ([0.2,0.5],[0.3,0.4]) & ([0.4,0.5],[0.1,0.2]) & ([0.3,0.6],[0.2,0.3]) & ([0.3,0.7],[0.1,0.3]) \\ ([0.2,0.7],[0.2,0.3]) & ([0.3,0.6],[0.2,0.4]) & ([0.4,0.7],[0.1,0.2]) & ([0.5,0.8],[0.1,0.2]) \\ ([0.5,0.6],[0.3,0.4]) & ([0.7,0.8],[0.1,0.2]) & ([0.5,0.6],[0.2,0.3]) & ([0.4,0.5],[0.3,0.4]) \\ ([0.3,0.6],[0.2,0.4]) & ([0.4,0.6],[0.2,0.3]) & ([0.1,0.4],[0.3,0.6]) & ([0.3,0.7],[0.1,0.2]) \\ ([0.6,0.7],[0.1,0.3]) & ([0.5,0.6],[0.3,0.4]) & ([0.5,0.6],[0.2,0.3]) & ([0.5,0.6],[0.2,0.4]) \end{pmatrix}.$$

In the follows, we can utilize the proposed procedure to get the most desirable alternative(s).

Step 1. We firstly determine fuzzy density of each decision maker, and its λ parameter. Suppose that $g(e_1) = 0.40$, $g(e_2) = 0.40$, $g(e_3) = 0.40$. Then λ of expert can be determined: $\lambda_1 = -0.44$. According to Eq. (3), we have $g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0.73$, $g(e_1, e_2, e_3) = 1$.

Step 2. By Eq. (6) or Definition 7, $\tilde{a}_{ij}^k$ is reordered such that $\tilde{a}_{ij}^{(k)} \leqslant \tilde{a}_{ij}^{(k+1)}$, then Utilize the generalized interval-valued intuitionistic fuzzy geometric aggregation operator

$$\begin{aligned} \tilde{a}_{ij} &= \mathrm{GIIFGA}_g(\tilde{a}_{ij}^1, \tilde{a}_{ij}^2, \tilde{a}_{ij}^3) \\ &= \Bigg(\left[\prod_{k=1}^{3}(a_{\tilde{a}_{ij}^{(k)}})^{\mu(A_{(k)})-\mu(A_{(k+1)})}, \prod_{k=1}^{3}(b_{\tilde{a}_{ij}^{(k)}})^{\mu(A_{(k)})-\mu(A_{(k+1)})}\right], \\ &\quad \left[1 - \prod_{k=1}^{3}(1 - c_{\tilde{a}_{ij}^{(k)}})^{\mu(A_{(k)})-\mu(A_{(k+1)})}, \right. \\ &\quad \left. 1 - \prod_{k=1}^{3}(1 - d_{\tilde{a}_{ij}^{(k)}})^{\mu(A_{(k)})-\mu(A_{(k+1)})}\right]\Bigg) \end{aligned}$$

to aggregate all the interval-valued intuitionistic fuzzy decision matrices $R^k = (\tilde{a}_{ij}^k)_{m \times n}$ $(k = 1, 2, 3)$ into a complex interval-valued intuitionistic fuzzy decision matrix $R = (\tilde{a}_{ij})_m \times n$ as follows:

$$R = \begin{pmatrix} ([0.3017, 0.4645], [0.2685, 0.3687]) & ([0.4373, 0.5650], [0.1282, 0.2983]) & ([0.2452, 0.4685], [0.3257, 0.4280]) & ([0.3299, 0.5720], [0.1911, 0.3925]) \\ ([0.3463, 0.5386], [0.2917, 0.3925]) & ([0.4353, 0.6715], [0.1683, 0.2983]) & ([0.4248, 0.6715], [0.1282, 0.2283]) & ([0.5310, 0.7083], [0.1343, 0.2616]) \\ ([0.5712, 0.7083], [0.1590, 0.2598]) & ([0.5720, 0.6732], [0.1590, 0.2598]) & ([0.5000, 0.6382], [0.1683, 0.3000]) & ([0.2751, 0.4356], [0.3257, 0.4622]) \\ ([0.3242, 0.4996], [0.2283, 0.3990]) & ([0.5000, 0.7083], [0.1282, 0.2616]) & ([0.1951, 0.4306], [0.2260, 0.3966]) & ([0.3366, 0.7000], [0.1000, 0.2000]) \\ ([0.6382, 0.7384], [0.1282, 0.2616]) & ([0.4685, 0.6075], [0.1716, 0.2982]) & ([0.5000, 0.6382], [0.2000, 0.3000]) & ([0.4685, 0.5720], [0.2614, 0.4280]) \end{pmatrix}.$$

Step 3. Since ([1,1],[0,0]) and ([0,0],[1,1]) are the largest and smallest interval-valued intuitionistic fuzzy values, respectively. For cost criteria c_1, c_4 and benefit criteria c_2, c_3, IV-IFPIS $\tilde{\alpha}^+$ and IV-IFNIS $\tilde{\alpha}^-$ can be simply denoted as follows:

$$\tilde{\alpha}^+ = (([0,0],[1,1])([1,1],[0,0])([1,1],[0,0])([0,0],[1,1])),$$
$$\tilde{\alpha}^- = (([1,1],[0,0])([0,0],[1,1])([0,0],[1,1])([1,1],[0,0])).$$

Denote the alternatives a_i $(i = 1,2,\ldots,5)$ by $x_i = (\tilde{a}_{i1}, \tilde{a}_{i2}, \tilde{a}_{i3}, \tilde{a}_{i4})$:

$$\begin{aligned} x_1 &= (([0.3017, 0.4645], [0.2685, 0.3687]), ([0.4373, 0.5650], [0.1282, 0.2982]), \\ &\quad ([0.2452, 0.4685], [0.3257, 0.4280]), ([0.3299, 0.5720], [0.1911, 0.3925])) \\ x_2 &= (([0.3463, 0.5386], [0.2917, 0.3925]), ([0.4353, 0.6715], [0.1683, 0.2983]), \\ &\quad ([0.4248, 0.6715], [0.1282, 0.2283]), ([0.5310, 0.7083], [0.1343, 0.2616])) \\ x_3 &= (([0.5712, 0.7083], [0.1590, 0.2598]), ([0.5720, 0.6732], [0.1590, 0.2598]), \\ &\quad ([0.5000, 0.6382], [0.1683, 0.3000]), ([0.2751, 0.4356], [0.3257, 0.4622])) \\ x_4 &= (([0.3242, 0.4996], [0.2283, 0.3990]), ([0.5000, 0.7083], [0.1282, 0.2616]), \\ &\quad ([0.1951, 0.4306], [0.2260, 0.3966]), ([0.3366, 0.7000], [0.1000, 0.2000])) \\ x_5 &= (([0.6382, 0.7384], [0.1282, 0.2616]), ([0.4685, 0.6075], [0.1716, 0.2982]), \\ &\quad ([0.5000, 0.6382], [0.2000, 0.3000]), ([0.4685, 0.5720], [0.2614, 0.4280])). \end{aligned}$$

Step 4. We determine fuzzy density of each criterion, and its λ parameter. Suppose that $g(c_1) = 0.40$, $g(c_2) = 0.25$, $g(c_3) = 0.37$, $g(c_4) = 0.20$, according to Eq. (4), the λ of criteria can be determined: $\lambda_2 = -0.44$. By Eq. (3), we have $g(c_1,c_2) = 0.60$, $g(c_1,c_3) = 0.70$, $g(c_1,c_4) = 0.56$, $g(c_2,c_3) = 0.68$, $g(c_2,c_4) = 0.43$, $g(c_3,c_4) = 0.54$, $g(c_1,c_2,c_3) = 0.88$, $g(c_1,c_2,c_4) = 0.75$, $g(c_2,c_3,c_4) = 0.73$, $g(c_1,c_3,c_4) = 0.84$, $g(c_1, c_2,c_3,c_4) = 1.0$.

Step 5. According to Eqs. (9) and (10), respectively, we calculate that

$$d_1(x_1, \tilde{\alpha}^+) = 0.5551, d_1(x_1, \tilde{\alpha}^-) = 0.5158,$$
$$d_2(x_2, \tilde{\alpha}^+) = 0.4836, d_2(x_2, \tilde{\alpha}^-) = 0.5827,$$
$$d_3(x_3, \tilde{\alpha}^+) = 0.5030, d_3(x_3, \tilde{\alpha}^-) = 0.5665,$$
$$d_4(x_4, \tilde{\alpha}^+) = 0.5217, d_4(x_4, \tilde{\alpha}^-) = 0.5196,$$
$$d_5(x_5, \tilde{\alpha}^+) = 0.5440, d_5(x_5, \tilde{\alpha}^-) = 0.5350.$$

Step 6. According to Eq. (11), we calculate the closeness coefficient of each alternative as follows:

$$r(x_1) = 0.4817, \quad r(x_2) = 0.5465, \quad r(x_3) = 0.5297,$$
$$r(x_4) = 0.4990, \quad r(x_1) = 0.4958.$$

Step 7. Rank all the alternatives a_i $(i = 1,2,\ldots,5)$ according to the closeness coefficients $r(x_i)$:

$$a_2 \succ a_3 \succ a_4 \succ a_5 \succ a_1.$$

Thus the most desirable alternative is a_2.

7. Conclusion

This study presents a multi-criteria interval-valued intuitionistic fuzzy group decision making method using Choquet integral-based TOPSIS, where interactions phenomena among the decision making criteria or preference of experts are taken into account. Being a generalization of intuitionistic fuzzy sets, the interval-valued intuitionistic fuzzy sets are suitable way to deal with uncertainty. In the evaluation process, we have developed a generalized intuitionistic fuzzy geometric aggregation operator which is shown that the GIIFGA operator generalizes both the IIFOWGA operator and IIFWGA operator. Then the GIIFGA operator is utilized to aggregate opinions of decision makers. After interval-valued intuitionistic fuzzy positive-ideal solution and interval-valued intuitionistic fuzzy negative-ideal solution were calculated based on Choquet integral-based Hamming distance, the relative closeness coefficients of alternatives were obtained and alternatives were ranked. Finally, an example has demonstrated the model is efficient and robust. The proposed procedure differs from previous approaches for multi-criteria group decision making not only due to the fact that the proposed method use interval-valued intuitionistic fuzzy set theory rather than intuitionistic fuzzy set or fuzzy set theory, which will not cause no any loss of information in the process of aggregation., but also due to the consideration the interactions phenomena among the decision making criteria or preference of experts, which approximates to the truth of real decision making problems. So it is quite good for real-world applications.

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