Nadir compromise programming: A model for optimization of multi-objective portfolio problem

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ABSTRACT

In problem of portfolio selection, financial Decision Makers (DMs) explain objectives and investment purposes in the frame of multi-objective mathematic problems which are more consistent with decision making realities. At present, various methods have introduced to optimize such problems. One of the optimization methods is the Compromise Programming (CP) method. Considering increasing importance of investment in financial portfolios, we propose a new method, called Nadir Compromising Programming (NCP) by expanding a CP-based method for optimization of multi-objective problems. In order to illustrate NCP performance and operational capability, we implement a case study by selecting a portfolio with 35 stock indices of Iran stock market. Results of comparing the CP method and proposed method under the same conditions indicate that NCP method results are more consistent with DM purposes.

1. Introduction

Markowitz (1952, 1959) presented the so-called ‘mean–variance’ model, which assumes that the total return of a portfolio can be described using the mean return of the assets and the variance of return (risk) between securities. The portfolios that offer the minimum risk for a given level of return form what it is called the efficient frontier (Fernández & Gómez, 2007). The mean–variance methodology of Markowitz (1952) for portfolio selection problem has been central to research activity and has served as a basis for the development of modern financial theory. Yet, the Markowitz mean–variance model disadvantages include:

- The Markowitz model was generally criticized as not efficient with axiomatic models of preferences for choice under risk (Bell, Raiffa, & Tversky, 1988).
- The Markowitz model was a quadratic model which finally was non-linear. Because his model was non-linear, so obtained results were often local optimum.
- The mean–variance model of Markowitz (1952) was the time needed to compute the covariance matrix from historical data especially when problems were large scale and the difficulty of solving the large scale quadratic programming problems. For example, in the literature, some algorithms such as those proposed by Konno (1990), Konno and Yamazaki (1991), Sharpe (1963, 1967), Elton, Gruber, and Padberg (1976) and Young (1998), are generated in order to linearize and improve the efficiency calculation of the Markowitz mean–variance model (Nawrocki & Carter, 1998; Shing & Nagasawa, 1999). Young (1998) established portfolio scientific management, by his researches. He introduced ‘sensitivity coefficient $\beta$', which measures stock volatility relative to the benchmark index or the capital market.

By considering Sharpe (1964)’s work and other present researches in the literature, it has been common in recent years to use coefficient $\beta$ as measurement criterion of risk. Make use of this coefficient, in order to facilitate financial portfolio management process, is based on “modern portfolio theory”.

By a certain classification, whole risk is divided into two controllable risk (unsystematic risk) and uncontrollable risk (systematic risk) parts. Also, according to “capital asset pricing model” (Sharpe, 1964), by making a various financial portfolio of market’s selected stocks, unsystematic risk becomes capable of being minimized. The systematic risk, which is derived of economic, politic, social and environmental changes of capital market, is uncontrollable and almost has the same process for different stocks. Evident to this claim is the same process of different indices price in time.

The coefficient $\beta$ is a criterion for systematic risk and can be an index for ranking different assets. If $\beta$ for an asset is more than 1, the asset’s return variations will be more than that of market and this asset is called ‘aggressive asset’. Vice versa, an asset with $\beta$ less than 1 means fewer variations than market variations. Similarly, this asset is called ‘defensive asset’.
Advantages of using coefficients $\beta$ as risk coefficients of each security include:

- Making the risk objective function, linear and using it in optimizing portfolio selection problems.
- Increase of securities number has no effect on risk objective and computational volume does not increase, so that it is in contrast to the Markowitz mean–variance model.
- Using coefficients $\beta$ as risk coefficients of each security, also consider market variations. Therefore in this paper, we will use coefficients $\beta$ as risk coefficients in modeling of a problem of multi-objective portfolio selection.

One of the optimization methods for multi-objective problems is the CP model. In the CP model, DMs are able to establish, easily and precisely, goal values of the considered objectives. Zeleny (1973, 1974) proposed the CP. Of advantages of CP model are its simplicity and its capability to be used in conditions which it is not possible to access goal values of objectives.

The CP model was introduced by Romero, Amador, and Barco (1987) in the financial literature. Also Ballestero and Romero (1973, 1974) proposed the CP. Of advantages of CP model are its simplicity and its capability to be used in conditions which it is not possible to access goal values of objectives.

The CP model was introduced by Romero, Amador, and Barco (1987) in the financial literature. Also Ballestero and Romero (1973, 1974) proposed the CP. Of advantages of CP model are its simplicity and its capability to be used in conditions which it is not possible to access goal values of objectives.

The CP model (Zeleny, 1974), only has been formulated on the basis of the nadir values. The aim of this paper is to propose a new model which called Nadir Compromise Programming (NCP) by developing the CP model which can be used to optimize multi-objective problems. Therefore, this paper continues as follow: in Section 2, we introduce the CP model and reformulate the CP model on the basis of the nadir values for each objective function in Section 3, which called NCP model. Then in Section 4, we illustrate the CP and NCP models on a sample of 35 stocks from the Iran stock exchange market by a multi-objective problem in portfolio selection under same conditions and then evaluate obtained results. Meanwhile we conclude this paper in Section 5.

2. The CP model

The CP model was proposed by Zeleny (1974). It consists of minimizing the distance between the achievement levels $f_k$ and the utopia values ($f_k^{\text{max}}$ or $f_k^{\text{min}}$) associated with each objective $k$. In the case that the more of the objective is better, the utopia values $f_k^{\text{max}}$ can be obtained as follows:

\[
f_k^{\text{max}} = \max f_k, \quad k = 1, 2, \ldots, K
\]

subject to:

\[
g_i(x) \leq b_i, \quad i = 1, 2, \ldots, m,
\]

$x \in S$

and the final model of CP by considering preference weights of objectives ($w_k$) can be formulated as follows:

\[
\begin{align*}
\min & \left\{ \sum_{k=1}^{K} w_k (\delta_k^{+})^p \right\}^\frac{1}{p} \\
\text{subject to:} & \quad f_k + \delta_k^{+} = f_k^{\text{max}}, \quad k = 1, 2, \ldots, K, \\
& \quad g_i(x) \leq b_i, \quad i = 1, 2, \ldots, m, \\
& \quad x \in S, \\
& \quad \delta_k^{+} \geq 0, \quad k = 1, 2, \ldots, K.
\end{align*}
\]  

The variable $\delta_k^{+}$ is deviational variable of surplus for constraint related to $k$th objective. Also if the case that less of the objective is better, the utopia values $f_k^{\text{min}}$ can be obtained as follows:

\[
f_k^{\text{min}} = \min f_k, \quad k = 1, 2, \ldots, K
\]

subject to:

\[
g_i(x) \leq b_i, \quad i = 1, 2, \ldots, m,
\]

$x \in S$

\] and the final model of CP by considering preference weights of objectives ($w_k$) can be formulated as follows:

\[
\begin{align*}
\min & \left\{ \sum_{k=1}^{K} w_k (\delta_k^{-})^p \right\}^\frac{1}{p} \\
\text{subject to:} & \quad f_k - \delta_k^{-} = f_k^{\text{min}}, \quad k = 1, 2, \ldots, K, \\
& \quad g_i(x) \leq b_i, \quad i = 1, 2, \ldots, m, \\
& \quad x \in S, \\
& \quad \delta_k^{-} \geq 0, \quad k = 1, 2, \ldots, K.
\end{align*}
\]  

The variable $\delta_k^{-}$ is deviational variable of surplus for constraint related to $k$th objective.

3. The NCP model

The CP model (Zeleny, 1974), only minimizes the distance between the achievement levels $f_k$ and the utopia values associated with each objective $k$. In here, we present a model on the base of the CP model and considering maximum distance from the nadir values of objectives ($w_k$) can be formulated as follows:

\[
\begin{align*}
\min & \quad y \\
\text{subject to:} & \quad y \geq \sum_{k=1}^{K} w_k (\delta_k^{+}) \\
& \quad f_k + \delta_k^{+} = f_k^{\text{max}}, \quad k = 1, 2, \ldots, K, \\
& \quad g_i(x) \leq b_i, \quad i = 1, 2, \ldots, m, \\
& \quad x \in S, \\
& \quad y \geq 0, \quad \delta_k^{+} \geq 0, \quad k = 1, 2, \ldots, K.
\end{align*}
\]  

where $y$ is always non-negative.
In Program (6), the objective is to minimize \( \delta_k^c \) and \( \delta_k^b \) in order to more closeness to the goal value of kth objective function.

(b) If objective is to minimize \( f_k \) on the basis of the nadir value \( k \), then we have:

\[
\text{Min } f_k \quad (\text{for } k = 1, 2, ..., K).
\]

If \( f_k \) (for \( k = 1, 2, ..., K \)) is the nadir value of kth objective, the constraint related to kth objective in the NCP model can be as follows:

\[
f_k \geq f_k.
\]

and finally the NCP model by considering preference weights of objectives \( w_k \), can be as follows:

\[
\min \left\{ \sum_{k=1}^{K} w_k (\delta_k^c)^p \right\}^{\frac{1}{p}}
\]

subject to:

\[
f_k - \delta_k^c = f_k, \quad k = 1, 2, ..., K,
\]

\[
x \in S,
\]

\[
f_k \in \mathbb{R}; \quad \delta_k^c, \delta_k^b \geq 0, \quad k = 1, 2, ..., K.
\]

In constraint \( f_k + \delta_k^c = f_k \) (for \( k = 1, 2, ..., K \)) of Program (7), the objective is to maximize \( \delta_k^c \) in order to more distance from the nadir value of kth objective function.

(c) If objective is to maximize \( f_k \) on the basis of the nadir value \( k \), then we have:

\[
\text{Max } f_k \quad (\text{for } k = 1, 2, ..., K)
\]

and the constraint related to this objective in the NCP model is:

\[
f_k \leq f_k.
\]

and finally the NCP model by considering preference weights of objectives \( w_k \), can be formulated as follows:

\[
\min \left\{ \sum_{k=1}^{K} w_k (-\delta_k^c)^p \right\}^{\frac{1}{p}}
\]

subject to:

\[
f_k + \delta_k^c = f_k, \quad k = 1, 2, ..., K,
\]

\[
x \in S,
\]

\[
\delta_k^c \geq 0, \quad k = 1, 2, ..., K.
\]

Also, in constraint \( f_k + \delta_k^c = f_k \) (for \( k = 1, 2, ..., K \)) of Program (8), the objective is to minimize \( \delta_k^c \) in order to more distance from the nadir value of kth objective function.

Generally, if we optimize A objective functions on the basis of their goal values and minimize B objective functions and maximize C objective functions, the final model of NCP can be written as follows:

\[
\min \left\{ \sum_{a=1}^{A} w_a (\delta_a^c + \delta_a^b)^p + \sum_{b=1}^{B} w_b (-\delta_b^c)^p + \sum_{c=1}^{C} w_c (-\delta_c^b)^p \right\}^{\frac{1}{p}}
\]

subject to:

\[
f_a - \delta_a^c = f_{a0}, \quad a = 1, 2, ..., A,
\]

\[
f_a + \delta_a^b = f_{a0}, \quad a = 1, 2, ..., A,
\]

\[
f_b + \delta_b^c = f_{b0}, \quad b = 1, 2, ..., B,
\]

\[
f_c - \delta_c^b = f_{c0}, \quad c = 1, 2, ..., C,
\]

\[
x \in S,
\]

\[
\delta_a^c, \delta_a^b \geq 0; \quad \delta_b^c \geq 0; \quad \delta_c^b \geq 0,
\]

\[
a = 1, 2, ..., A; \quad b = 1, 2, ..., B; \quad c = 1, 2, ..., C,
\]

\[
f_{a0} \in \mathbb{R}, \quad a = 1, 2, ..., A.
\]

\[
(9)
\]

where \( \sum_{a=1}^{A} w_a + \sum_{b=1}^{B} w_b + \sum_{c=1}^{C} w_c = 1 \) (\( w_a, w_b, w_c > 0 \)), for \( a = 1, 2, \ldots, A \) and \( b = 1, 2, \ldots, B \) and \( c = 1, 2, \ldots, C \).

4. Numerical example

We illustrate our developed model through a portfolio selection problem where several conflicting objectives are considered. We consider a sample of 35 stocks from the Iran stock exchange. The data and observations (from June 23, 2003 to June 29, 2008) of the in-sample period are used as the training set to determine the models parameters and specifications.

The evaluation process consists of the following steps:

- The definition of the objectives.
- The determination, for each objective, of the utopia and nadir values and the goal values.
- The formulation of, and subsequently the solution to the CP (min–max) and NCP (min–max) models. At the end of this section, we will present and discuss the results obtained from both models.

In portfolio selection problem, the DM can consider several conflicting objectives. The objectives adopted by Markowitz (1952) are mean and variance. Lee and Chesser (1980) and Zopounidis et al. (1999) propose a set of objectives that the DM can consider to evaluate the stocks. For the illustration purposes, we will consider the following objectives:

- The first linear objective \( \left( \sum_{j=1}^{n \beta_j x_j} \right) \), is risk objective function.

- The second linear objective \( \left( E \left( \sum_{j=1}^{n \beta_j x_j} \right) = \sum_{j=1}^{n \mu_j x_j} \right) \), is expected rate of return objective function. The rate of return \( \left( r_j = \frac{P_{j,t}-P_{j,t-1}+D_{j,t}(P_{j,t-1})}{P_{j,t-1}} \right) \) measures the profitability of the stock where the income can be in the form of random capital gain and dividend. Here \( P_j \) is the price of the stock \( j \) at time \( t \) and \( D_{j,t} \) is the dividend received during the period \( [t-1,t] \). \( r_j \) is random and normally distributed with known mean \( \mu_j \) and variance \( \sigma_j^2 \). This objective is to be maximized.

- In here our proposed objective is the third linear objective \( \left( \sum_{j=1}^{n \beta_j x_j} \right) \), namely initial cost of investment objective function. In real world, many people suffer because they have not enough money for secure investments. Thus the aim this is which they spend less money while will obtain their favorite results from other objectives. Here \( P_j \) is the price of stock \( j \) (with known formal currency) in the last under study day. Let \( N \) be total number of present stocks in the optimum portfolio. Therefore, the initial cost of investment objective function can be obtained without considering the value \( N \) as follows:

\[
Z = P_1(Nx_1) + P_2(Nx_2) + \cdots + P_{35}(Nx_{35})
\]

\[
\Rightarrow Z = N(P_1x_1 + P_2x_2 + \cdots + P_{35}x_{35})
\]

\[
\Rightarrow Z = N \left( \sum_{j=1}^{35} P_j x_j \right) = \frac{Z}{N} = \frac{N \left( \sum_{j=1}^{35} P_j x_j \right)}{N}
\]

\[
\Rightarrow Z = \frac{N}{N} = \sum_{j=1}^{35} P_j x_j.
\]

(10)
The main portfolio selection problem can be formulated as follows:

\[
\begin{align*}
\text{opt} & \quad f_1 = \sum_{j=1}^{35} p_j x_j \\
\text{max} & \quad f_2 = \sum_{j=1}^{35} \mu_j x_j \\
\text{min} & \quad f_3 = \sum_{j=1}^{35} p_j x_j \\
\end{align*}
\]

subject to: \( x_1 + x_2 + x_3 + x_{15} = 0.2 \), \( x_5 + x_6 = 0.2 \), \( x_1 + x_6 = 0.2 \), \( x_4 + x_6 + x_{17} + x_{18} + x_{19} + x_{20} + x_{21} + x_{22} + x_{23} = 0.2 \), \( x_9 + x_{19} + x_{11} + x_{12} + x_{13} + x_{14} + x_{24} + x_{25} + x_{26} + x_{27} + x_{28} + x_{29} + x_{30} + x_{31} + x_{32} + x_{33} + x_{34} + x_{35} = 0.2 \), \( 0 \leq x_j \leq 0.1 \), \( j = 1, 2, \ldots, 35 \).

(11)

The Program (11) is transformed to a NCP model, \( f_2 \), and \( f_3 \), are the nadir values associated with the expected return and the initial cost objectives in solution space of Program (11), respectively. The Beta goal is equal to 1. In order to use the NCP model, we consider \( I_{\infty} \) metric for the NCP model. On this basis Program (11) can be reformulated as follows:

\[
\begin{align*}
\text{min} & \quad y \\
\text{subject to:} & \quad y \geq w_1 (\delta^+_1 + \delta^-_1), \\
& \quad y \geq w_2 (\delta^+_2 - \delta^-_2), \\
& \quad y \geq w_3 (\delta^+_3 - \delta^-_3) \\
& \quad \begin{cases} f_1 - \delta^+_1 = 1, \\
& \quad f_1 + \delta^-_1 = 1, \\
& \quad f_2 - \delta^+_2 = f_2, \\
& \quad f_2 + \delta^-_2 = f_2, \\
& \quad x_1 + x_2 + x_3 + x_{15} = 0.2, \\
& \quad x_5 + x_6 = 0.2, \\
& \quad x_1 + x_6 = 0.2, \\
& \quad x_4 + x_{16} + x_{17} + x_{18} + x_{19} + x_{20} + x_{21} + x_{22} + x_{23} = 0.2, \\
& \quad x_9 + x_{19} + x_{11} + x_{12} + x_{13} + x_{14} + x_{24} + x_{25} + x_{26} + x_{27} + x_{28} + x_{29} + x_{30} + x_{31} + x_{32} + x_{33} + x_{34} + x_{35} = 0.2, \\
& \quad 0 \leq x_j \leq 0.1, \quad j = 1, 2, \ldots, 35. 
\end{cases}
\end{align*}
\]

(12)

The objectives considered, in this example, are the risk \( \beta \), the expected return and the initial cost of investment. They are equally weighed (\( w_1 = w_2 = w_3 = 1/3 \)).

In Table 1 we present data concerning the different securities of the Iran stock market for the years 2003 to 2008. These data are available in Table 1. The five columns of the Table 1 are the stock number, the stocks, the risk \( \beta \), the expected rate of return and the stock price in the last exchanged day, respectively.

The utopia and nadir values of the expected return and the initial cost objectives have been represented in Table 2. The goal fixed for the risk \( \beta \) is equal to 1. The software of Lingo 8.0 has been used to solve the mathematical programs of the CP (min–max) and NCP (min–max) models under same conditions.

In Table 2, we present the optimal proportion of investment in each stock for two selected portfolios on the basis of the CP (min–max) and NCP (min–max) models. We notice that the results in portfolio of the CP (min–max) model are close to those obtained.
could be said that in here, this more consistence is product of modeling of the nadir value of each objective in the NCP method.

5. Conclusion

In spite of the fundamentality of the Markowitz mean–variance model in optimal portfolio selection problem, always there have been critics because of being quadratic and making use of covariance matrix in the risk objective of this model. So it was seen that using Beta risk coefficients can be a proper replacement in order to model investment risk objective function. In the other section of this paper, we presented the CP model and proposed the NCP model which can optimize multi-objective problems. The NCP model was formulated on the basis of the nadir values of each objective. Then we introduced a multi-objective problem to select optimal portfolio in Iran stock market and optimized the multi-objective problem by two the CP (min–max) and NCP (min–max) models under the same conditions. The obtained results confirmed that in spite of being feasible and optimal of two portfolios, the NCP (min–max) model can satisfy DM purposes better.

References


