ELSEVIER

European Journal of Operational Research 80 (1995) 410–417

EUROPEAN JOURNAL OF OPERATIONAL RESEARCH

Theory and Methodology

Using AHP for resource allocation problems

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Received March 1993; revised June 1993

Abstract

AHP has been used for solving multi-criteria resource allocation problems by converting them into equivalent single objective, maximization-type LP problems. At least two approaches can be identified for such applications. In the first approach, the AHP priorities are used as coefficients in the objective function of the LP format, and in the second approach, the benefit–cost ratios are used as the coefficients. This paper evaluates the two approaches. It is shown here that both the approaches are suitable if the criteria (used in the AHP model) are sought to be maximized. However, they are shown to be inappropriate if the criteria are sought to be minimized. A new, simple approach is suggested for the latter case and an extension has been proposed to tackle problems with mixed criteria. The implications of the study on other preference elicitation methods have also been pointed out.

Keywords: Resource allocation; Multiple criteria; AHP; Linear programming

1. Introduction

Multi-criteria resource allocation (mcra) problems involve allocation of limited resources to different activities keeping in mind many conflicting criteria. They have been effectively solved using multi-criteria decision making (mcdm) techniques. The Analytic Hierarchy Process (AHP) of Saaty (1980) has emerged as a useful decision making technique for solving mcra problems (Wedley, 1990). In AHP applications to resource allocation the mcra problem is converted into an equivalent single objective maximization-type LP problem and at least two approaches can be identified from a detailed study of the literature.

One approach is to use AHP priorities as coefficients in a single objective, maximization-type LP problem (Saaty and Mariano, 1979; Liberatore, 1987; Weiss and Rao, 1986). In this approach, the objective function specifies the expected priority which has to be maximized subject to a given constraint set. This approach will be referred to as the Expected Priority (EP) approach throughout this paper.

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SSDI 0377-2217(93)E0240-X

A mathematical statement of the EP approach is as follows:

$$\text{Max } \sum p_i x_i \tag{1}$$

subject to a given constraint set

where p_i is the priority, and x_i is the allocation to be made to the activity (or alternative) i.

The other approach is based on the benefit-cost ratios of each activity (Saaty and Kearns, 1985). In this, the benefits and costs of allocating resources to specific activities are separately obtained, in the form of priorities, using two different AHP models. The ratios are used as coefficients in a single objective, maximization-type LP format.

A mathematical statement of the benefit-cost approach (hereafter referred to as the BC approach) can be provided as follows:

$$\text{Max } \sum (b_i/c_i) x_i \tag{2}$$

subject to a given constraint set

where b_i is the priority representing benefits from, and c_i is the priority representing costs of, allocating resources to activity i.

Both the EP and BC approaches provide effective means of converting an mcra problem into an equivalent single objective, maximization-type LP problem. However there does not seem to be any study in the literature for evaluating these approaches. Such an evaluation is provided in this paper.

A key result of the study is that the EP and BC approaches do not guarantee correct solutions. They provide correct results for criteria which are sought to be maximized (hereafter referred to as 'direct criteria'). However, when the criteria are sought to be minimized (hereafter referred to as 'inverse criteria') they do not guarantee correct results. We suggest that in general AHP-based resource allocation problems should be converted into equivalent double-objective function problems for obtaining correct results.

2. Methodology of evaluation

The main aim of this paper is to evaluate the two approaches against a reference approach. Such a reference is possible only when a single, quantitative criterion is considered, and when linear utilities are assumed. Hence we use this framework for proving our arguments. We have provided logical generalizations for more complex cases based upon the results of the single, quantitative criterion framework.

For this framework AHP priorities can be obtained through either the ratio method or through pairwise comparison matrices. Use of the ratio method requires the assumption of linear utilities (Wedley, 1990). We make this assumption initially for the purpose of direct comparison with the reference approach, because under this assumption, there exists a definite relation between the actual measures of performance and the corresponding coefficients of the EP and BC approaches. This relation is explained for both the approaches in the next section.

For both the approaches, priorities of qualitative criteria can be obtained through pairwise comparison matrices. If the assumption of linear utilities is not valid, priorities for the quantitative criteria should also be obtained through pairwise comparisons only.

The performance of the approaches under the assumption of non-linear utilities will be inferred qualitatively from that under the linear utility assumption.

2.1. The EP and BC approaches

Let a_i (where $a_i > 0$) represent the actual measure of performance of an alternative 'i' with respect to a criterion. If the criterion is a direct one, in the EP approach, using the ratio method, the priority of the alternative is given by

$$p_i = a_i / \sum a_i. \tag{3}$$

For an inverse criterion, the priorities are obtained through normalization of the reciprocals of performance measures as

$$p_i = (1/a_i) / \sum (1/a_i). \tag{4}$$

In the BC approach, when only one direct criterion is considered, the approach shown in (2) is modified as

$$\text{Max } \sum b_i x_i \tag{5}$$

subject to the given constraint set

and the benefit derived from allocating resources to a particular activity is estimated as

$$b_i = a_i / \sum a_i. \tag{6}$$

Similarly, when only one inverse criterion is considered, (2) becomes

$$\text{Max } \sum (1/c_i) x_i$$

subject to the given constraint set (7)

and the 'cost' of allocation can be estimated as

$$c_i = a_i / \sum a_i. \tag{8}$$

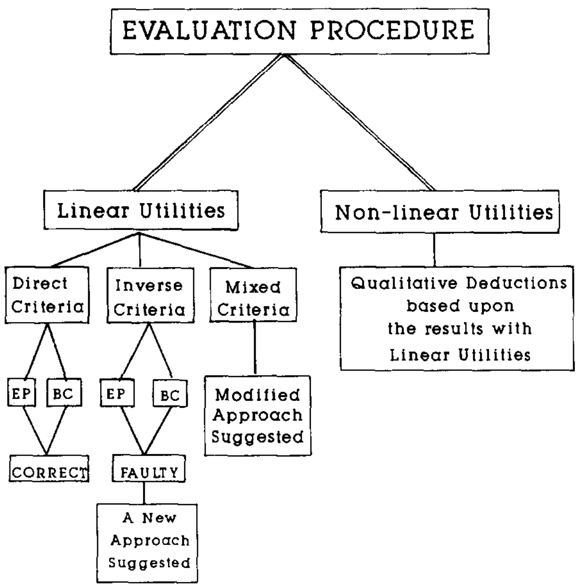

Fig. 1. Logic of evaluation of the EP and BC approaches.

2.2. The reference approach

Using the performance measures a_i, a single objective LP problem can be written as

$$\text{Optimize } \sum a_i x_i \tag{9}$$

subject to a given constraint set.

This problem can be solved using conventional LP techniques. It has to be maximized for direct criteria and minimized for inverse criteria. This LP problem forms the reference approach in this paper. The optimal solution of this reference problem will be compared with the results obtained by using the EP and BC approaches, as they also aim to provide only the same optimal results. It is again stressed here that the reference approach provides correct results only under the assumption of linear utilities. This approach cannot be employed for the case of non-linear utilities since disproportionate relations exist between actual measures 'a_i' and the priorities of (or, preferences for) alternatives obtained using AHP in such a case.

The performance of the EP and BC approaches vary depending upon whether the criterion is a direct one or an inverse one. The next two sections deal respectively with these cases. The methodology of comparison is pictorially depicted in Fig. 1.

3. Evaluation for direct criteria

When only direct criteria are considered, both the EP and BC approaches provide correct results. The necessary proofs will be provided in the next two subsections. For the purpose of proofs an obvious result will be made use of, which can be stated as: "The optimal solution(s) of a single objective LP problem will not change if all the objective function coefficients are multiplied by the same positive constant α." This result can be very easily proved and is not shown here.

3.1. The EP approach

Theorem 1 proves that, for direct criteria, the EP approach provides the same results as the reference approach.

Theorem 1. *For a single, quantitative, and direct criterion, under the assumption of linear utilities, problems* (1) *and* (9) *provide the same optimal solutions.*

Proof. For a single, quantitative and direct criterion, and under the assumption of linear utilities, the priorities p_i of (1) can be obtained using (3) as

$$p_i = a_i / \sum a_i = \alpha a_i$$

where $\alpha = 1/\Sigma a_i$ = a positive constant, and a_i are the coefficients used in (9). Thus all the coefficients p_i of (1) are the result of multiplying the corresponding coefficients a_i of (9) with the same positive constant α. Hence, problems (1) and (9) have the same optimal solution(s).

3.2. The BC approach

The BC approach also provides the same results as that of the reference method as proved in Theorem 2.

Theorem 2. *For a single, quantitative, and direct criterion, under the assumption of linear utilities, problems* (5) *and* (9) *provide the same optimal solutions.*

Proof. For a single, quantitative and direct criterion, under the assumption of linear utilities, the benefits b_i of (5) can be obtained using (6) as

$$b_i = a_i / \sum a_i = \alpha a_i.$$

The rest of the proof is similar to that of Theorem 1.

Thus in the case of direct criteria, the conversion of an AHP-based mcra problem into an equivalent single-objective LP problem is valid. This result is based upon the assumption of linear utilities. If we relax this assumption, we do not have any reference method to compare the solutions. However, as an extrapolation of the results under the assumption of linear utilities, it can be expected that the two approaches will provide valid solutions in the case of non-linear utilities also.

4. Evaluation for inverse criteria

Both the EP and BC approaches do not always provide optimal solutions when inverse criteria are used. This is because both the approaches convert the inverse criterion into a direct criterion involving preferences and use these preference priorities in a maximization-type LP problem.

4.1. The EP approach

When (4) is used for priority computation in (1) for inverse criteria, the resulting solution will not always be the same as that of the reference approach. We illustrate the inconsistency using two simple examples.

Reference approach	EP approach
Example 1:	
Min $2x + 3y + 5z$	Max $ax + by + cz$
subject to:	subject to the same
$x + y \leq 10$	set of constraints.
$2x + 3y \geq 7$	
$x + y + z \geq 6$ and	
$x, y, z \geq 0$	
Optimal solution: **(6, 0, 0)**.	Solution: **Unbounded.**
Minimum objective	Objective function
function value: **12.**	value: **Infinite.**
Example 2:	
Min $2x + 3y + 5z$	Max $ax + by + cz$
subject to:	subject to the same
$2x + y \leq 10$	set of constraints.
$x + 3z \leq 7$	
$x + 2y + z \geq 6$ and	
$x, y, z \geq 0$	

Optimal solution: **(0, 3, 0)**. Minimum objective function value: **9**.	Solution: **(0, 10, 2.333)**. Objective function value (using cost coefficients of the original problem): **41.66**, where: $a = (\frac{1}{2})/[(\frac{1}{2}) + (\frac{1}{3}) + (\frac{1}{5})] = \frac{15}{31}$, $b = \frac{10}{31}$ and $c = \frac{6}{31}$.

The inconsistency may be due to the fact that although the priorities (using inverse criteria) here preserve ordinality of the preference structure of activities, they may not always preserve cardinality. Hence, when an application warrants only ordinality of alternatives, such as when choosing one among different alternatives or when ranking different alternatives, priority computation using (4) is acceptable even for inverse criteria. But, for resource allocation problems within the context of an LP format, cardinality also becomes important and hence using such priorities does not guarantee optimal results. This is a rather straightforward result in LP methodology, but as mentioned earlier the focus here is on the AHP to LP conversion process used in the EP approach.

4.2. The BC approach

This approach provides the same solution as EP approach as proved in Theorem 3.

Theorem 3. *For a single, quantitative, and inverse criterion, under the assumption of linear utilities, problems* (7) *and* (1) *provide the same solution.*

Proof.

The coefficient of the activity x in (7)

$$= \frac{1}{c_i} = \frac{1}{a_i / \sum a_i} \quad \text{using (8)}$$

$$= \sum a_i * \left(\frac{1}{a_i}\right) = \sum a_i * \sum\left(\frac{1}{a_i}\right) * \left(\frac{1}{a_i}\right) / \sum\left(\frac{1}{a_i}\right) = \beta * \left(\frac{1}{a_i}\right) / \sum\left(\frac{1}{a_i}\right)$$

$$= \beta * p_i \text{ (from (4) – for inverse criterion)}$$

where $\beta = \Sigma a_i * \Sigma(1/a_i)$, is a positive constant, and p_i is the coefficient of activity x_i in (1). Thus all coefficients of activities in (7) are multiplied by the positive constant β to obtain the coefficients of (1). Hence, both (7) and (1) provide the same optimal solution(s). Hence the theorem is proved. Hence the BC approach also does not guarantee optimal results.

Thus both the EP and BC approaches do not guarantee optimal solutions for the case of a single, inverse criterion. This result can be extended for the case of non-linear utilities also. For example, when all the constraints are lower bounded, maximization of the EP or BC objective function will always result in an unbounded solution, irrespective of whether it is a case of linear utility or not.

4.3. The suggested approach

As stated earlier, the cardinality of the preference structure of the alternatives becomes important for resource allocation, and the inverse criterion should not be converted into a corresponding direct criterion involving preferences. Thus, any approach which preserves the cardinal structure and at the same time uses the inverse criterion directly in the LP problem can be used to arrive at correct results. The cardinality is preserved when the degree of *non-preference* is computed from the performance measures 'a_i'. Thus the suggested approach uses (3) for priority computation. In such a case, the coefficients 'p_i' represent a measure of dis-utility and may be called *inverse priorities* as against the term *priorities*. These inverse priorities are the same as the 'costs' as defined in (8), but the authors do not use this term here because in their opinion the term is not appropriate. The resulting expected inverse priority objective function can be *minimized* (an appropriate treatment for inverse criteria) using conventional LP methods. Now the modified problem can be written as

$$\text{Min} \sum p_i' x_i \tag{10}$$

subject to the constraint set

where p_i' = inverse priorities $= a_i/(\Sigma a_i)$. Theorem 4 proves that this approach provides correct results.

Theorem 4. *For a single, quantitative, inverse criterion, under the assumption of linear utilities, problems* (10) *and* (9) *provide the same optimal solutions.*

Proof. Similar to that for Theorem 1.

As this approach provides correct results for the linear utility assumption, we can expect correct results for the case of non-linear utilities also.

For cases when the pairwise comparison matrices are to be used, the inverse priorities for minimization problems should be obtained through suitably designed questionnaires. Thus for cost (minimization) criterion, while priorities are obtained through questions of the type

"*As far as cost is concerned, alt. X is (equally/moderately more ...) preferred to alt. Y*",

inverse priorities should be obtained through questions of the form

"*As far as cost is concerned, alt. X is (equally/moderately more ...) difficult to purchase than alt. Y*".

The above analysis has dealt with only quantitative criteria. The case of inverse, qualitative criteria has not been considered here because such a straightforward derivation of priorities is not possible in that case, and consequently a straightforward proof for establishing the correctness of any RA method with these criteria is also not possible. However, the results of the analysis involving an inverse, quantitative criterion can be usefully employed for this case also. Thus it is necessary to derive the inverse-priorities corresponding to any inverse criterion (qualitative/quantitative) and use the inverse-priorities in a minimizing objective function for getting correct results.

4.4. Implications for other preference elcitation methods

While the discussion in the preceding subsections have concentrated only on AHP, the findings can be applied to other preference elicitation methods as well (Keeney and Raiffa, 1976; Goicochea et al., 1982). Thus, while dealing with preferences for inverse criteria, it may be worthwhile to elicit the dis-utilities (or, inverse preferences) of alternatives and employ these dis-utilities appropriately in any analysis such as resource allocation. For example, maximization of the expected utilities can be employed for direct criteria, while minimization of expected dis-utilities should be used for inverse criteria.

5. The most general case

The most general case involves optimization of many objective functions, some of which are to be maximized while the rest are to be minimized. For this problem, both the EP and BC approaches are not appropriate as they have been shown to be faulty while handling inverse criteria. A modified approach is given below.

(a) Consider all the direct criteria separately and synthesize final priorities (p_i) of activities on the basis of these criteria only.

(b) Consider all the inverse criteria separately and synthesize the final inverse priorities (p_i') of activities on the basis of these criteria only.

Now, the following double objective function problem will be obtained:

$$\text{Max} \sum p_i x_i,$$

$$\text{Min} \sum p_i' x_i$$

subject to the constraint set.

This double objective function problem or the bicriteria problem can be solved using any of the multi-objective mathematical programming approaches (Goicochea et al., 1982) such as the simple weighted additive procedure or goal programming, to provide satisficing solutions.

6. Summary and conclusions

Two approaches, viz., the expected priority approach and the benefit–cost ratio approach, involved in using the Analytic Hierarchy Process for resource allocation problems have been examined in this paper. Under some simplifying assumptions, both the approaches have been proved to give correct results when only direct criteria are considered. It has been illustrated that they need not provide optimum solutions when only inverse criteria are considered. On the basis of these findings, it is concluded in this paper that both the approaches are not appropriate for resource allocation problems, when many direct and inverse criteria have to be simultaneously considered. A new and simple approach has been suggested to obtain correct results for dealing with inverse criteria. On the basis of this new approach, a double objective function methodology has been proposed for the case involving many, mixed criteria. The implications of these findings on other preference elicitation methods have also been pointed out. It is hoped that the findings of the paper will provide theoretical insight, and the alternative approaches suggested will be useful, for utilizing AHP in resource allocation applications.

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