

Scientific Computation

Thomas H. Pulliam  
David W. Zingg

# Fundamental Algorithms in Computational Fluid Dynamics

 Springer

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 Springer

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*To Harv Lomax and Joe Steger*

# Preface

The field of computational fluid dynamics (CFD) has matured substantially over the past 30 years and has proven its worth in numerous areas of science and engineering. Although there are different numerical algorithms that can be used to solve the equations governing fluid flow, the key algorithmic concepts that provide the foundations of CFD are by now well established. The purpose of this book is to give a detailed and comprehensive description of some important and widely used algorithms. While the field of CFD will continue to evolve, the algorithms described in this book will continue to provide a basis for understanding most numerical approaches to the solution of the Euler and Navier-Stokes equations in the foreseeable future.

This book is intended to be used as a textbook, that is, within the context of a course in CFD. It is suitable as either a first or a second course at the senior undergraduate or graduate level. As a result of the popularity of CFD, the number of engineers and scientists using it has greatly increased. Many of these users will not have a graduate-level education in CFD; hence the field is increasingly being covered in the undergraduate curricula. It is important to recognize that even a user (as opposed to a developer) of CFD needs some exposure to the underlying theory of both fluid dynamics and numerical algorithms in order to make intelligent use of CFD. The material in this book is equally appropriate to both users and developers of computational methods for fluid dynamics.

In a sense, this is a sequel to our previous book, *Fundamentals of Computational Fluid Dynamics*, written with Harvard Lomax. Whereas that book deals primarily with simple model equations, this one concentrates on the Euler and Navier-Stokes equations. The two books can be used quite naturally in a two-course sequence.<sup>1</sup> However, the present book can also be used as a first course in CFD, as long as the students have had some previous exposure to basic numerical methods. [Chapter 2](#) provides a concise summary of some of the key ideas in our earlier book. Our emphasis is again on a detailed treatment of specific core topics rather than a comprehensive treatment of the entire field. Moreover, our focus is on mature algorithms as opposed to those currently under development.

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<sup>1</sup> They have been used in this manner at the University of Toronto for many years.

Given our emphasis on depth as opposed to breadth, several important topics are deferred. Examples include spatial discretizations on unstructured grids, finite-element and spectral methods, and turbulence models, which are left to authors with greater experience in these particular subjects. The algorithms presented here form the basis of several important flow solvers and have been used for countless computations. While many other worthy algorithms exist, understanding and programming the algorithms primarily emphasized in this book should provide the reader with a basis for understanding virtually any algorithm in use today.

A key feature of this text is the use of examples and programming assignments based on the one-dimensional and quasi-one-dimensional Euler equations. Of course, these equations omit important physics (viscosity, heat conduction, and turbulence) and numerics (factorization, meshing). Nevertheless, a great deal can be learned from implementing and studying the algorithms in this context, and the assignments are feasible within a typical one-term course. In order to derive the full benefit of this text, the reader is encouraged to complete the programming assignments associated with each chapter.

We present this book in the hope that it will contribute to an intelligent and creative approach to the development and application of CFD.

Moffett Field, August 2013  
Toronto

Thomas H. Pulliam  
David W. Zingg

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